

# GKM THEORY, CHARACTERISTIC CLASSES AND THE EQUIVARIANT COHOMOLOGY RING OF THE REAL GRASSMANNIAN

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ABSTRACT. We use GKM theory to understand equivariant cohomology of real Grassmannian and oriented Grassmannian, then confirm Casian&Kodama's conjecture on Borel description which says the ring generators are equivariant Pontryagin classes, Euler classes in even dimension, and one more new type of classes in odd dimension. All these Grassmannians are equivariantly formal, hence many results for equivariant cohomology have counterparts for ordinary cohomology. We also give additive basis in terms of characteristic polynomials and canonical classes, and calculate Poincaré series and characteristic numbers.

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## 1. INTRODUCTION

The study of homology and cohomology of real and complex Grassmannian was initially based on Ehresmann's decomposition [Eh37] in terms of Schubert cells. The relations between Schubert classes and characteristic

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classes were given by Chern [Ch48] for real Grassmannians in  $\mathbb{Z}/2$  coefficients. Using the techniques of spectral sequences of fibre bundles, Borel [Bo53] gave a unified way to describe the mod 2 cohomology of homogeneous spaces in terms of characteristic classes.

These pioneering work applies not only to ordinary cohomology but also to equivariant cohomology. For compact connected Lie group  $G$  and its connected closed subgroup  $H$  of the same rank, the maximal torus  $T$  acts on the left of the homogeneous space  $G/H$  with the Borel description of its equivariant cohomology in  $\mathbb{Q}$  coefficients:

$$H_T^*(G/H) = \mathrm{St}^* \otimes_{(\mathrm{St}^*)^{W_G}} (\mathrm{St}^*)^{W_H}$$

where  $\mathrm{St}^*$  is the symmetric algebra of the dual Lie algebra  $\mathfrak{t}^*$ , and  $W_G, W_H$  are the Weyl groups of  $G$  and  $H$ .

For example, if we consider complex flag varieties as quotients of unitary groups  $U(n)$ , then the Weyl groups are products of symmetric groups and the Weyl group invariants  $(\mathrm{St}^*)^{W_G}, (\mathrm{St}^*)^{W_H}$  consist of symmetric polynomials in appropriate variables. In geometric terminology, these invariants are polynomials of equivariant Chern classes of canonical bundles, with relations from Whitney product formula.

Similarly, we can apply the Borel description to even dimensional oriented Grassmannian viewed as quotients of  $SO(n)$ , of which Weyl groups are semi-products of symmetric groups and groups of power of 2. Using invariant theory, we can obtain its equivariant cohomology ring generated on equivariant Pontryagin classes and equivariant Euler classes of canonical bundle and complementary bundle, with relations from Whitney product formula and square formula of Euler class as top Pontryagin class. Notice that oriented Grassmannian is a natural 2-fold cover of real Grassmannian, then we can identify equivariant cohomology of real Grassmannian as  $\mathbb{Z}/2$ -invariant of equivariant cohomology of oriented Grassmannian. Due to the lack of preferred orientation for subspace in  $\mathbb{R}^n$ , there is no Euler class of canonical bundle or complementary bundle over real Grassmannian. These facts give even dimensional real Grassmannian its Borel description of equivariant cohomology ring generated on equivariant Pontryagin classes, with relations from Whitney product formula.

The above reasoning breaks down in the case of odd dimensional oriented Grassmannian  $SO(2n+2)/SO(2k+1) \times SO(2n-2k+1)$  in which  $SO(2n+2)$  and  $SO(2k+1) \times SO(2n-2k+1)$  have different ranks of maximal tori. In this paper, we will use the techniques developed by Goresky, Kottwitz and MacPherson to overcome the gap between even and odd dimensional Grassmannians and derive the same Borel description. Contrary to the aforementioned direction from oriented Grassmannian to real Grassmannian, we will first work out cohomology of real Grassmannian confirming a conjecture by Casian&Kodama [CK13], then proceed to cohomology of oriented Grassmannian. Together with the ring structure, we will give additive basis, compute Pioncaré series and characteristic numbers, and try to relate these results with Schubert calculus on real Grassmannian.

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## 2. TORUS ACTIONS, EQUIVARIANT COHOMOLOGY

In this section, we recall the basics of torus action, equivariant cohomology. Our major reference is All-day&Puppe [AP93].

**2.1. Torus actions and isotropy weights at fixed points.** Throughout the paper, a manifold  $M$  is always assumed to be smooth, compact, but not necessarily oriented nor connected. Let torus  $T$  act on a manifold  $M$ , we will denote  $M^T$  as the fixed-point set. For any point  $p$  in a connected component  $C$  of  $M^T$ , there is the **isotropy representation** of  $T$  on the tangent space  $T_p M$ , which splits into weight spaces  $T_p M = V_0 \oplus V_{[\alpha_1]} \oplus \cdots \oplus V_{[\alpha_r]}$  where the non-zero distinct weights  $[\alpha_1], \dots, [\alpha_r] \in \mathfrak{t}_{\mathbb{Z}}^*/\pm 1$  are determined only up to signs. Comparing with the tangent-normal splitting  $T_p M = T_p C \oplus N_p C$ , we get that  $T_p C = V_0$  and  $N_p C = V_{[\alpha_1]} \oplus \cdots \oplus V_{[\alpha_r]}$ . Since  $N_p C = V_{[\alpha_1]} \oplus \cdots \oplus V_{[\alpha_r]}$  is of even dimension, the dimensions of  $M$  and components of  $M^T$  will be of the same parity. If  $\dim M$  is even, the smallest possible components of  $M^T$  could be isolated points. If  $\dim M$  is odd,

the smallest possible components of  $M^T$  could be isolated circles. Moreover, since  $T$  acts on the normal space  $N_p C$  by rotation, this gives the normal space  $N_p C$  an orientation.

For any subtorus  $K$  of  $T$ , we get two more actions automatically: the **sub-action** of  $K$  on  $M$  and the **residual action** of  $T/K$  on  $M^K$ .

Let  $E$  be a complex or real vector bundle over  $M$ . We say  $E$  is a  **$T$ -equivariant vector bundle** over  $M$ , if

- (1) Both  $E$  and  $M$  have  $T$ -actions
- (2)  $t \in T$  maps  $E_p$  to  $E_{t \cdot p}$  linearly, for any  $p \in M$ .

**2.2. Equivariant cohomology.** Let torus  $T^n$  act on a manifold  $M$ . The  $T$ -equivariant cohomology of  $M$  is defined using the Borel construction  $H_T^*(M) = H^*((ET \times M)/T)$ , where  $ET = (S^\infty)^n$  with  $BT = ET/T = (\mathbb{C}P^\infty)^n$  and the coefficient of cohomology will usually be  $\mathbb{Q}$  throughout the paper, unless otherwise mentioned to be  $\mathbb{Z}$ . By this definition, if we denote  $\mathfrak{t}^*$  as the Lie dual algebra of  $T$ , then  $H_T^*(pt) = H^*(ET/T) = H^*((\mathbb{C}P^\infty)^n) = \mathbb{S}\mathfrak{t}^*$  is a polynomial ring  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$  or  $\mathbb{Z}[\alpha_1, \dots, \alpha_n]$  under the identification that  $c_1(\gamma_i \rightarrow BT) = \alpha_i$ , where  $\gamma_i \rightarrow BT$  is the canonical complex line bundle on the  $i$ -th  $\mathbb{C}P^\infty$ -component of  $BT$  and  $\alpha_i \in \mathfrak{t}_\mathbb{Z}^*$  is the integral weight dual to the  $i$ -th  $S^1$ -component of  $T$ . The trivial map  $\iota : M \rightarrow pt$  induces a homomorphism  $\iota^* : H_T^*(pt) \rightarrow H_T^*(M)$  and hence makes  $H_T^*(M)$  a  $H_T^*(pt)$ -module.

For  $T$ -equivariant complex vector bundle  $V$  over  $M$ , we can define the equivariant Chern classes and Euler classes as  $c^T(V) = c((ET \times V)/T)$ ,  $e^T(V) = e((ET \times V)/T) \in H_T^*(M, \mathbb{Z})$ . For  $T$ -equivariant real vector bundle  $W$  over  $M$ , we can define the equivariant Pontryagin classes as  $p^T(W) = p((ET \times W)/T) \in H_T^*(M, \mathbb{Z})$ .

The famous Atiyah-Bott-Berline-Vergne(ABBV) localization formula says:

**Theorem 2.2.1** (Atiyah-Bott-Berline-Vergne Localization Theorem, [BV83, AB84]). *On oriented  $T$ -manifold  $M$ , any equivariant cohomology class  $\omega \in H_T^*(M)$  can be integrated as*

$$\int_M \omega = \sum_{C \subseteq M^T} \int_C \frac{\omega|_C}{e^T(NC)}$$

where the summation is taken for every component  $C \subseteq M^T$  with normal bundle  $NC$  and equivariant Euler class  $e^T(NC)$ .

Inspired by this localization theorem, we can hope for more connections between the manifold  $M$  and its fixed-point set  $M^G$ , if  $H_T^*(M)$  is actually a free  $H_T^*(pt)$ -module.

**Definition 2.2.1.** An action of  $T$  on  $M$  is **equivariantly formal** if  $H_T^*(M)$  is a free  $H_T^*(pt)$ -module.

Using the techniques of spectral sequences, equivariant formality has various equivalent expressions.

**Theorem 2.2.2** (Equivalences of equivariant formality, [AP93]). *Let torus  $T$  act on manifold  $M$ , the following conditions about equivariant cohomology are equivalent:*

- (1) *The  $T$ -action is equivariantly formal, i.e.  $H_T^*(M)$  is a free  $H_T^*(pt)$ -module*
- (2) *The Leray-Serre sequence of the fibration  $M \hookrightarrow (M \times ET)/T \rightarrow BT$  collapses with  $E_\infty = E_2 = H^*(BT) \otimes H^*(M)$*
- (3)  *$H_T^*(M) \cong H_T^*(pt) \otimes H^*(M)$  as  $H_T^*(pt)$ -module*
- (4)  *$H_T^*(M) \rightarrow H^*(M)$ , defined as the restriction to the fibre  $M$ , is surjective*
- (5) *Any additive basis of  $H^*(M)$  can be lifted to  $H_T^*(M)$ , hence gives an additive  $H_T^*(pt)$ -basis for  $H_T^*(M)$*
- (6)  $\sum \dim H^*(M^T) = \sum \dim H^*(M)$

*Remark 2.2.1.* The equivalences among (2)(3)(4)(5) are a direct application of Leray-Hirsch theorem (which works not only for  $\mathbb{Q}$  coefficients, but also for  $\mathbb{Z}$  coefficients). For the equivalence to the remaining conditions (1)(6) in  $\mathbb{Q}$  coefficients, see Allday&Puppe [AP93].

*Remark 2.2.2.* When the Betti numbers of  $M$  and  $M^T$  are known, the equality  $\sum \dim H^*(M^T) = \sum \dim H^*(M)$  is a handy way to verify the equivariant formality.

*Remark 2.2.3.* The fibre inclusion  $M \hookrightarrow (M \times ET)/T$  induces a homomorphism  $H_T^*(M) \rightarrow H^*(M)$  factoring through  $\mathbb{Q} \otimes_{H_T^*(pt)} H_T^*(M)$ , where  $\mathbb{Q}$  has a  $H_T^*(pt) = \mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -algebra structure from the constant-term morphism  $\mathbb{Q}[\alpha_1, \dots, \alpha_n] \rightarrow \mathbb{Q} : f(\alpha_1, \dots, \alpha_n) \mapsto f(0)$ . When the  $T$ -action is equivariantly formal, i.e.  $H_T^*(M) \cong H_T^*(pt) \otimes_{\mathbb{Q}} H^*(M)$ , then we can recover  $H^*(M)$  as  $H^*(M) \cong \mathbb{Q} \otimes_{H_T^*(pt)} H_T^*(M)$ .

### 3. GKM THEORIES

In this section, we recall the Chang-Skjelbred lemma, the GKM condition and the author's recent work on generalized GKM theories for possibly non-orientable manifolds in both even and odd dimensional cases.

**3.1. Chang-Skjelbred lemma and GKM condition.** For every point  $p \in M$ , its stabilizer is defined as  $T_p = \{t \in T \mid t \cdot p = p\}$ , and its orbit is  $\mathcal{O}_p = T/T_p$ . Let's set the 1-skeleton  $M_1 = \{p \mid \dim \mathcal{O}_p \leq 1\}$ . If  $H_T^*(M)$  is a free  $H_T^*(pt)$ -module, i.e. equivariantly formal, Chang and Skjelbred [CS74] proved that  $H_T^*(M)$  only depends on the fixed-point set  $M^T$  and the 1-skeleton  $M_1$ .

**Theorem 3.1.1** (Chang-Skjelbred Lemma, [CS74]). *If torus  $T$  acts on  $M$  equivariantly formally, then the following short sequence is exact:*

$$0 \longrightarrow H_T^*(M) \longrightarrow H_T^*(M^T) \longrightarrow H_T^{*+1}(M_1, M^T)$$

This short exact sequence enables one to describe the equivariant cohomology  $H_T^*(M)$  as a sub-ring of  $H_T^*(M^T)$ , subject to certain algebraic relations determined by the 1-skeleton  $M_1$ . To apply the Chang-Skjelbred Lemma, we will follow Goresky, Kottwitz and MacPherson's idea to start with the smallest-dimensional 0-skeleton  $M^T$  and 1-skeleton  $M_1$ .

**Definition 3.1.1** (GKM condition). An action of torus  $T$  on  $M$  is **GKM** if

- (1) The fixed-point set  $M^T$  consists of isolated points or isolated circles.
- (2) At each fixed point  $p \in M^T$ , the non-zero weights  $[\alpha_1], \dots, [\alpha_n] \in \mathfrak{t}_{\mathbb{Z}}^*/\pm 1$  of the isotropy  $T$ -representation  $T \curvearrowright T_p M$  are pair-wise independent.

*Remark 3.1.1.* As mentioned in the Subsection 2.1, the dimensions of  $M$  and of  $M^T$  have the same parity. The (1) in the GKM condition 3.1.1 means that  $M^T$  consists of isolated points when  $M$  is even dimensional or  $M^T$  consists of isolated circles when  $M$  is odd dimensional.

*Remark 3.1.2.* The GKM condition is equivalent to requiring the 1-skeleton  $M_1$  to be 2-dimensional when  $M$  is even dimensional or 3-dimensional when  $M$  is odd dimensional.

*Remark 3.1.3.* If  $M$  has a  $T$ -invariant stable almost complex structure, then at each fixed point, the isotropy weights  $\alpha_1, \dots, \alpha_n \in \mathfrak{t}_{\mathbb{Z}}^*$  are determined without ambiguity of signs.

**3.2. Generalized GKM theory: the even dimensional case.** Goresky, Kottwitz and MacPherson [GKM98] considered torus actions on algebraic varieties when the fixed-point set  $M^T$  is finite and the 1-skeleton  $M_1$  is a union of spheres  $S^2$ . They proved that the cohomology  $H_T^*(M)$  can be described in terms of congruence relations on a graph determined by the 1-skeleton  $M_1$ . Their ideas actually work even in non-orientable case where the 1-skeleton  $M_1$  could have  $\mathbb{R}P_{[\alpha]}^2$  components for some weights  $[\alpha] \in \mathfrak{t}_{\mathbb{Z}}^*/\pm 1$ .

**Definition 3.2.1** (Generalized GKM graph in even dimension). The **GKM graph** of a GKM action of torus  $T$  on  $M^{2n}$  consists of

**Vertices:** There are two types of vertices

•: for each fixed point in  $M^T$

**Empty dot:** for each exceptional orbit in a  $\mathbb{R}P_{[\beta]}^2 \in M_1$

**Edges & Weights:** A solid edge with weight  $[\alpha]$  for each  $S^1_{[\alpha]}$  joining two  $\bullet$ 's representing its two fixed points, and a dotted edge with weight  $[\beta]$  for each  $\mathbb{R}P^2_{[\beta]}$  joining a  $\bullet$  to an empty vertex.

**Theorem 3.2.1** (Generalized GKM theorem in even dimension, [He16b]). *If the action of torus  $T$  on a (possibly non-orientable) manifold  $M^{2n}$  is equivariantly formal and GKM, then we can denote its GKM graph as  $\Gamma$ , with vertex set  $V = M^T$  and weighted edge set  $E$ , moreover the equivariant cohomology has a graphic description*

$$H_T^*(M) = \{f : V \rightarrow \mathbb{S}t^* \mid f_p \equiv f_q \pmod{\alpha} \text{ for each solid edge } \overline{pq} \text{ with weight } \alpha \text{ in } E\}$$

*Remark 3.2.1.* Since  $H_T^*(\mathbb{R}P^2_{[\alpha]}) = H_T^*(pt)$  does not contribute to congruence relations, we only need to use the  $S^2_{[\beta]}$  components in  $M_1$  as weighted edges to construct an **effective** GKM graph, which does not necessarily have the same number of edges for every vertex.

**3.3. Generalized GKM theory: the odd dimensional case.** In odd dimension, we can also follow Goresky, Kottwitz and MacPherson's idea of considering smallest-dimensional 1-skeleton, i.e.  $M_1$  is 3-dimensional. Then according to Chang-Skjelbred lemma, the GKM theory boils down to equivariant cohomology of  $S^1$ -equivariantly formal 3-manifolds, studied in [He16a].

Similar to the even dimensional case, we can construct a GKM graph for each odd dimensional  $T$ -manifold under GKM condition.

**Definition 3.3.1** (GKM graph in odd dimension). The **GKM graph** for a GKM action of torus  $T$  on (possibly non-orientable) manifold  $M^{2n+1}$  consists of

**Vertices:** There will be two types of vertices.

◦ for each fixed circle  $C \subset M^T$ .

◻ for each 3d connected component  $N^3_{[\alpha]}$  in  $M^{T_{[\alpha]}}$  of some codimension-1 subtorus  $T_{[\alpha]}$  which has Lie algebra  $\mathfrak{t}_\alpha \subset \mathfrak{t}$  annihilated by  $\alpha$ . The ◻ is then weighted with  $[\alpha]$ .

**Edges:** An edge joins a (◻,  $N$ ) to a (◦,  $C$ ), if the 3d manifold  $N$  contains the fixed circle  $C$  and hence is a connected component of  $M^{T_\alpha}$  for an isotropy weight  $[\alpha]$  of  $C$ . There are no edges directly joining ◦ to ◦, nor ◻ to ◻.

In order to derive a GKM-type graphic description of  $H_T^*(M^{2n+1})$ , we need to fix in advance an orientation  $\theta_i$  for each fixed circle  $C_i \subseteq M^T$ , and also fix an orientation for each orientable  $M^{T_{[\alpha]}} \subseteq M_1$ .

**Theorem 3.3.1** (Generalized GKM theorem in odd dimension, [He16b]). *If an action of torus  $T$  on (possibly non-orientable) manifold  $M^{2n+1}$  is equivariantly formal and GKM, and let's denote its GKM graph as  $\Gamma$ , with two types of vertex sets  $V_\circ$  and  $V_\square$  and edge set  $E$ , then an element of the equivariant cohomology  $H_T^*(M)$  can be written as:*

$$(P, Q\theta) : V_\circ \longrightarrow \mathbb{S}t^* \oplus \mathbb{S}t^*\theta$$

where  $\theta$  is the generator of  $H^1(S^1)$ , under the relations that for each ◻ representing a 3d component  $N$  of some  $M^{T_\alpha}$  and the neighbour ◦'s representing the fixed circles  $C_1, \dots, C_k$  on this component,

- if  $N$  is non-orientable,

$$P_{C_1} \equiv P_{C_2} \equiv \dots \equiv P_{C_k} \pmod{\alpha}$$

- if  $N$  is orientable,

$$P_{C_1} \equiv P_{C_2} \equiv \dots \equiv P_{C_k} \text{ and } \sum_{i=1}^k \pm Q_{C_i} \equiv 0 \pmod{\alpha}$$

where the sign for each  $Q_{C_i}$  is specified by comparing the predetermined orientation  $\theta_i$  with the induced orientation of  $N$  on  $C_i$ .

*Remark 3.3.1.* As discussed in [He16b], different choices of orientations on  $C_i \subseteq M^T$  and orientable  $M^{T_{[\alpha]}} \subseteq M_1$  give the isomorphic equivariant cohomology. When  $M$  has a  $T$ -invariant stable almost complex structure, then the isotropy weights  $\alpha$  can be determined without ambiguity of signs. Moreover,  $M^T$  and  $M^{T_\alpha} \subseteq M_1$  are equipped with induced stable almost complex structures, hence are oriented canonically.

#### 4. EQUIVARIANT COHOMOLOGY RING OF COMPLEX GRASSMANNIAN

In this section, we recall the GKM description and Borel description of equivariant cohomology ring of complex Grassmannian, together with the characteristic basis and canonical basis of the additive structure. We use the notation  $G_k(\mathbb{C}^n)$  for the Grassmannian of  $k$ -dimensional complex subspaces in  $\mathbb{C}^n$ .

**4.1. GKM description of complex Grassmannian.** As shown by Guillemin, Holm and Zara [GHZ06], for compact connected group  $G$  and its closed connected subgroup  $H$  of the same rank as  $G$ , the homogeneous space  $G/H$  is GKM and equivariantly formal under the left action of maximal torus  $T$ , hence has a graphic description for its equivariant cohomology. For example, the GKM graph of  $T^n \curvearrowright U(n)/(U(k) \times U(n-k))$  is the Johnson graph  $J(n, k)$ , of which each vertex is a  $k$ -element subset  $S \in \{1, \dots, n\}$  and two vertices  $S, S'$  are joined by an edge if they differ by one element. For later use in the case of real Grassmannian, we will give an explicit construction for the GKM graph of complex Grassmannian.

**Proposition 4.1.1** (1-skeleton of complex Grassmannian). *The  $T^n$ -action on  $G_k(\mathbb{C}^n)$  has  $\binom{n}{k}$  fixed points of the form  $\oplus_{i \in S} \mathbb{C}_i$ , where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$  and  $\mathbb{C}_i$  is the  $i$ -th component of  $\mathbb{C}^n$ . The isotropy weights at  $\oplus_{i \in S} \mathbb{C}_i$  are  $\{\alpha_j - \alpha_i \mid i \in S, j \notin S\}$ , and join  $\oplus_{i' \in S} \mathbb{C}_{i'}$  to  $\oplus_{i' \in (S \setminus \{i\}) \cup \{j\}} \mathbb{C}_{i'}$  via  $\{(\oplus_{i' \in S \setminus \{i\}} \mathbb{C}_{i'}) \oplus L \mid L \in \mathbb{P}(\mathbb{C}_i \oplus \mathbb{C}_j)\} \cong \mathbb{C}P^1$  in the 1-skeleton.*

*Proof.*  $T^n$  acts on  $\mathbb{C}^n$  linearly by  $(t_1, \dots, t_n) \cdot (z_1, \dots, z_n) = (t_1 z_1, \dots, t_n z_n)$  and hence induces an action on  $G_k(\mathbb{C}^n)$  by mapping every  $k$ -dimensional subspace  $V$  to  $t \cdot V$  for each  $t \in T^n$ . A fixed point  $V$  of the  $T$ -action on  $G_k(\mathbb{C}^n)$  is exactly a  $k$ -dimensional sub-representation of  $\mathbb{C}^n = \sum_{i=1}^n \mathbb{C}_i$ . Since  $\sum_{i=1}^n \mathbb{C}_i$  has distinct weights  $\alpha_1, \dots, \alpha_n$ , a  $k$ -dimensional sub-representation is of the form  $\oplus_{i \in S} \mathbb{C}_i$  for  $S$ , a  $k$ -element subset of  $\{1, 2, \dots, n\}$ .

To understand the isotropy weights at  $\oplus_{i \in S} \mathbb{C}_i \in G_k(\mathbb{C}^n)$ , notice that the tangent space of  $G_k(\mathbb{C}^n)$  at  $\oplus_{i \in S} \mathbb{C}_i$  is  $\text{Hom}_{\mathbb{C}}(\oplus_{i \in S} \mathbb{C}_i, \oplus_{j \notin S} \mathbb{C}_j) \cong (\oplus_{i \in S} \mathbb{C}_i)^* \otimes_{\mathbb{C}} (\oplus_{j \notin S} \mathbb{C}_j) \cong \oplus_{i \in S} \oplus_{j \notin S} (\mathbb{C}_i^* \otimes_{\mathbb{C}} \mathbb{C}_j)$  with pair-wise independent weights  $\{\alpha_j - \alpha_i \mid i \in S \text{ and } j \notin S\}$ .

The finiteness of fixed points and pair-wise independence of isotropy weights at every fixed point verifies that the  $T^n$  action on  $G_k(\mathbb{C}^n)$  is GKM. Moreover, for a  $k$ -element subset  $S \subseteq \{1, 2, \dots, n\}$  and a weight  $\alpha_j - \alpha_i$  with  $j \notin S, i \in S$ , the 2-sphere

$$\left\{ \left( \bigoplus_{i' \in S \setminus \{i\}} \mathbb{C}_{i'} \right) \oplus L \mid L \in \mathbb{P}(\mathbb{C}_i \oplus \mathbb{C}_j) \right\} \cong \mathbb{C}P^1$$

connects the  $T$ -fixed point  $\oplus_{i \in S} \mathbb{C}_i$  with the  $T$ -fixed point  $\oplus_{i' \in (S \setminus \{i\}) \cup \{j\}} \mathbb{C}_{i'}$ , and is fixed by the corank-1 subtorus  $T_{\alpha_j - \alpha_i}$  of which Lie algebra is  $\text{Ker}(\alpha_j - \alpha_i)$ .  $\square$

We have seen previously that the  $T^n$  action on  $G_k(\mathbb{C}^n)$  is equivariantly formal in  $\mathbb{Z}$  coefficients and of course in  $\mathbb{Q}$  coefficients. Then we can apply the even dimensional GKM theorem to  $G_k(\mathbb{C}^n)$  using congruence relations on the Johnson graph  $J(n, k)$ .

**Theorem 4.1.1** (GKM description of complex Grassmannian, [GZ03]). *Let  $\mathcal{S}$  be the collection of  $k$ -element subsets of  $\{1, 2, \dots, n\}$ , then the equivariant cohomology of the  $T^n$  action on  $G_k(\mathbb{C}^n)$  is*

$$H_T^*(G_k(\mathbb{C}^n), \mathbb{Q}) = \{f : \mathcal{S} \rightarrow \mathbb{Q}[\alpha_1, \dots, \alpha_n] \mid f_S \equiv f_{S'} \pmod{\alpha_j - \alpha_i} \text{ for } S, S' \in \mathcal{S} \text{ with } S \cup \{j\} = S' \cup \{i\}\}$$

Using Morse theory on graphs, Guillemin and Zara constructed additive basis for equivariant cohomology of GKM manifolds.

**Theorem 4.1.2** (Canonical basis of complex Grassmannian [GZ03]). *There is a self-indexing Morse function on  $\mathcal{S}$*

$$\phi : \mathcal{S} \longrightarrow \mathbb{R} : S \longmapsto 2\left(\sum_{i \in S} i\right) - k(k+1)$$

and a canonical class  $\tau_S \in H_T^{\phi(S)}(G_k(\mathbb{C}^n), \mathbb{Q})$  for each  $S \in \mathcal{S}$  such that

- (1)  $\tau_S$  is supported upward, i.e.  $\tau_S(S') = 0$  if  $\phi(S') \leq \phi(S)$

(2)  $\tau_S(S) = \prod'(\alpha_j - \alpha_i)$  where the product is taken over the weights at  $S$  connecting to  $S'$  with  $\phi(S') < \phi(S)$

Moreover,  $\{\tau_S, S \in \mathcal{S}\}$  gives an additive basis of  $H_T^*(G_k(\mathbb{C}^n), \mathbb{Q})$ .

*Remark 4.1.1.* As shown in [GZ03], these canonical classes  $\tau_S$  are exactly the equivariant Schubert classes via the relation that if  $S$  consists of the elements  $i_1 < i_2 < \dots < i_k$ , then the corresponding Schubert symbol is  $(i_1 - 1, i_2 - 2, \dots, i_k - k)$ .

*Remark 4.1.2.* In this paper, we only need the existence of canonical classes  $\tau_S$ . The general formula of  $\tau_S$  restricted at each fixed point was given by Guillemin&Zara [GZ03] and simplified by Goldin&Tolman [GT09].

**4.2. Borel description of complex Grassmannian.** Besides the canonical basis in terms of equivariant Schubert classes, equivariant characteristic classes and characteristic polynomials on complex Grassmannian will give ring generators and additive basis for its cohomology ring.

**Theorem 4.2.1** (Borel description of complex Grassmannian, see Bott&Tu [BT82]). *For the complex Grassmannian  $G_k(\mathbb{C}^n)$ , let  $c_1, c_2, \dots, c_k$  and  $\bar{c}_1, \bar{c}_2, \dots, \bar{c}_{n-k}$  be the Chern classes of the canonical bundle on  $G_k(\mathbb{C}^n)$  and its complementary bundle respectively, then*

$$H^*(G_k(\mathbb{C}^n), \mathbb{Z}) = \frac{\mathbb{Z}[c_1, c_2, \dots, c_k; \bar{c}_1, \bar{c}_2, \dots, \bar{c}_{n-k}]}{(1 + c_1 + c_2 + \dots + c_k)(1 + \bar{c}_1 + \bar{c}_2 + \dots + \bar{c}_{n-k}) = 1}$$

The relation  $(1 + c_1 + c_2 + \dots + c_k)(1 + \bar{c}_1 + \bar{c}_2 + \dots + \bar{c}_{n-k}) = 1$  makes either  $c_1, c_2, \dots, c_k$  or  $\bar{c}_1, \bar{c}_2, \dots, \bar{c}_{n-k}$  as the ring generators of the cohomology  $H^*(G_k(\mathbb{C}^n), \mathbb{Z})$ . Certain monomials of  $c_1, c_2, \dots, c_k$  actually give an additive basis of the cohomology  $H^*(G_k(\mathbb{C}^n), \mathbb{Z})$ , stated by Carrell [Ca78] for complex Grassmannians in  $\mathbb{C}$  coefficients as a result of “standard combinatorial reasoning”. Later, details of proof were supplied by Jaworowski [Ja89] for real Grassmannians in  $\mathbb{Z}/2$  coefficients which can be adapted to complex Grassmannians in  $\mathbb{Z}$  coefficients.

**Theorem 4.2.2** (Characteristic basis of complex Grassmannian, [Ca78, Ja89]). *The set of monomials  $c_1^{r_1} c_2^{r_2} \dots c_k^{r_k}$  of cohomological degree  $2d = \sum_{i=1}^k 2ir_i$  satisfying the condition  $\sum_{i=1}^k r_i \leq n - k$  forms an additive basis for  $H^{2d}(G_k(\mathbb{C}^n), \mathbb{Z}), 0 \leq d \leq k(n - k)$ .*

The complex Grassmannian  $G_k(\mathbb{C}^n)$ , viewed as the homogeneous space  $U(n)/(U(k) \times U(n - k))$ , has a natural action of  $U(n)$  on the left, hence also has an action of the maximal torus  $T^n$ . Notice that the cohomology  $H^*(G_k(\mathbb{C}^n), \mathbb{Z})$  does not have odd-degree elements, the Leray-Serre sequence of  $G_k(\mathbb{C}^n) \hookrightarrow ET \times_T G_k(\mathbb{C}^n) \rightarrow BT$  collapses at  $E_2 = H^*(BT, \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(G_k(\mathbb{C}^n), \mathbb{Z})$  with  $H_T^*(G_k(\mathbb{C}^n), \mathbb{Z}) \cong H^*(BT, \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(G_k(\mathbb{C}^n), \mathbb{Z})$  as  $H^*(BT, \mathbb{Z})$ -modules. Therefore, the action  $T^n \curvearrowright G_k(\mathbb{C}^n)$  is equivariantly formal in  $\mathbb{Z}$  coefficients, and of course in  $\mathbb{Q}$  coefficients. However,  $H_T^*(G_k(\mathbb{C}^n), \mathbb{Z}) \cong H^*(BT, \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(G_k(\mathbb{C}^n), \mathbb{Z})$  is only a  $H^*(BT, \mathbb{Z})$ -module isomorphism which

- (1) neither gives the  $H^*(BT, \mathbb{Z})$ -algebra structure of  $H_T^*(G_k(\mathbb{C}^n), \mathbb{Z})$
- (2) nor specifies a map  $H_T^*(G_k(\mathbb{C}^n), \mathbb{Z}) \rightarrow H^*(BT, \mathbb{Z}) \otimes_{\mathbb{Z}} H^*(G_k(\mathbb{C}^n), \mathbb{Z})$

The above two problems can be resolved using equivariant versions of the Borel description which is usually given for a compact connected Lie group  $G$  with maximal torus  $T$  and a closed connected subgroup  $H$  containing  $T$  as

$$H_T^*(G/H) = \text{St}^* \otimes_{(\text{St}^*)^{W_G}} (\text{St}^*)^{W_H}$$

where  $W_G$  and  $W_H$  are the Weyl groups of  $G$  and  $H$ . For the complex Grassmannian  $G_k(\mathbb{C}^n) = U(n)/(U(k) \times U(n - k))$ , we have the Weyl group  $W_G = S_n$ , the symmetric group of  $n$  elements, and the Weyl group  $W_H = S_k \times S_{n-k}$ . Under these Weyl group actions, it is well known that the invariant elements in  $\text{St}^*$  are symmetric polynomials, or geometrically the equivariant Chern classes (for instance see [Tu10]):

**Theorem 4.2.3** (Equivariant Borel description of complex Grassmannian). *For the complex Grassmannian  $G_k(\mathbb{C}^n) = U(n)/(U(k) \times U(n - k))$ , let  $T^n$  be the maximal torus of  $U(n)$  which acts on the left of  $G_k(\mathbb{C}^n)$ , and*

$\alpha_1, \alpha_2, \dots, \alpha_n$  be the integral basis for its Lie dual algebra  $\mathfrak{t}^*$ , also let  $c_1^T, c_2^T, \dots, c_k^T$  and  $\bar{c}_1^T, \bar{c}_2^T, \dots, \bar{c}_{n-k}^T$  be the equivariant Chern classes of the canonical bundle on  $G_k(\mathbb{C}^n)$  and its complementary bundle respectively, then

$$H_T^*(G_k(\mathbb{C}^n), \mathbb{Z}) = \frac{\mathbb{Z}[\alpha_1, \alpha_2, \dots, \alpha_n][c_1^T, c_2^T, \dots, c_k^T; \bar{c}_1^T, \bar{c}_2^T, \dots, \bar{c}_{n-k}^T]}{c^T \bar{c}^T = \prod_{i=1}^n (1 + \alpha_i)}$$

Since the equivariant Chern classes  $c_1^T, c_2^T, \dots, c_k^T$  and  $\bar{c}_1^T, \bar{c}_2^T, \dots, \bar{c}_{n-k}^T$  lifts the ordinary Chern classes  $c_1, c_2, \dots, c_k$  and  $\bar{c}_1, \bar{c}_2, \dots, \bar{c}_{n-k}$ , we get

**Theorem 4.2.4** (Equivariant characteristic basis of complex Grassmannian). *The set of monomials  $(c_1^T)^{r_1} (c_2^T)^{r_2} \dots (c_k^T)^{r_k}$  satisfying the condition  $\sum_{i=1}^k r_i \leq n - k$  forms an additive  $H_T^*(pt)$ -basis for  $H_T^*(G_k(\mathbb{C}^n), \mathbb{Z})$ .*

*Proof.* Combine the ordinary characteristic basis with the equivalence (5) of equivariant formality.  $\square$

**4.3. Relations between the Borel and GKM descriptions.** Since the characteristic monomials  $(c^T)^I = (c_1^T)^{i_1} \dots (c_k^T)^{i_k}$  in Borel description and the canonical classes in GKM description are both basis for the free  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -module  $H_T^*(G_k(\mathbb{C}^n), \mathbb{Q})$ , there will be transformations  $K, \bar{K}$  between them such that

$$(c^T)^I = \sum_S K_S^I \tau_S$$

$$\tau_S = \sum_I \bar{K}_I^S (c^T)^I$$

where  $K_S^I, \bar{K}_I^S \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$ .

*Remark 4.3.1.* In this paper, we only the existence of the transformations  $K, \bar{K}$ . For the flag manifold  $Fl(\mathbb{C}^n)$ , Kaji gave explicit algorithms on how to decide the polynomials  $K_S^I, \bar{K}_I^S$ . It would also be interesting to know what the  $K_S^I, \bar{K}_I^S$  explicitly are for complex Grassmannian.

The Littlewood-Richardson rule for equivariant Schubert classes is that there are polynomials  $N_{S,S'}^{S''} \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$  (See Knutson&Tao [KT03]) such that

$$\tau_S \tau_{S'} = \sum_{S''} N_{S,S'}^{S''} \tau_{S''}$$

On the other hand, the multiplication of the equivariant characteristic classes is straightforward. We can express the total equivariant Chern classes  $c^T, \bar{c}^T$  of the canonical bundle  $\gamma$  and its complementary bundle  $\bar{\gamma}$  in GKM description at each fixed point  $\oplus_{i \in S} \mathbb{C}_i \in G_k(\mathbb{C}^n)$ . Note that the canonical bundle  $\gamma$ , complementary bundle  $\bar{\gamma}$  and tangent bundle  $TG_k(\mathbb{C}^n)$  restricted at  $\oplus_{i \in S} \mathbb{C}_i \in G_k(\mathbb{C}^n)$  for a  $k$ -element subset  $S \subset \{1, \dots, n\}$  are the vector spaces  $\oplus_{i \in S} \mathbb{C}_i$ ,  $\oplus_{j \notin S} \mathbb{C}_j$  and  $\oplus_{i \in S} \oplus_{j \notin S} (\mathbb{C}_i^* \otimes \mathbb{C}_j)$  respectively, we get

$$c^T|_S = c^T(\gamma|_S) = \prod_{i \in S} (1 + \alpha_i)$$

$$\bar{c}^T|_S = c^T(\bar{\gamma}|_S) = \prod_{j \notin S} (1 + \alpha_j)$$

$$e^T|_S = e^T(T_S G_k(\mathbb{C}^n)) = \prod_{i \in S} \prod_{j \notin S} (\alpha_j - \alpha_i)$$

Since  $\gamma \oplus \bar{\gamma} = \sum_{i=1}^n \mathbb{C}_i$ , this also shows why there is the relation  $c^T \bar{c}^T = \prod_{i=1}^n (1 + \alpha_i)$ . If we denote  $e_l(x_1, \dots, x_m)$  as the  $l$ -th elementary polynomial in variables  $x_1, \dots, x_m$ , then  $c_l^T|_S = e_l(\alpha_{i \in S}), \bar{c}_l^T|_S = e_l(\alpha_{j \notin S})$ .



Together with the ABBV localization formula 2.2.1, we can calculate the equivariant Chern numbers (see [Tu10])

$$\begin{aligned} \int_{G_k(\mathbb{C}^n)} (c^T)^I &= \sum_S \frac{((c_1^T)^{i_1} \cdots (c_k^T)^{i_k})|_S}{e_S^T} \\ &= \sum_S \frac{e_1^{i_1}(\alpha_{i \in S}) \cdots e_k^{i_k}(\alpha_{i \in S})}{\prod_{i \in S} \prod_{j \notin S} (\alpha_j - \alpha_i)} \in \mathbb{Q}[\alpha_1, \dots, \alpha_n] \end{aligned}$$

When the cohomological degree of characteristic polynomial matches with the dimension of Grassmannian, we then get a formula for the ordinary Chern numbers by substituting  $\alpha_i = i$ :

$$\int_{G_k(\mathbb{C}^n)} c^I = \sum_S \frac{e_1^{i_1}(S) \cdots e_k^{i_k}(S)}{\prod_{i \in S} \prod_{j \notin S} (j - i)} \in \mathbb{Q}$$

where the above two formulas are taking sums for all  $k$ -element subsets  $S \subset \{1, \dots, n\}$ .

*Remark 4.3.2.* The above characteristic numbers are with respect to the characteristic classes of the canonical bundle and complementary bundle not the tangent bundle. However,

$$c^T(T(G_k(\mathbb{C}^n)))|_S = c^T(\oplus_{i \in S} \oplus_{j \notin S} (\mathbb{C}_i^* \otimes \mathbb{C}_j)) = \prod_{i \in S} \prod_{j \notin S} (1 + \alpha_j - \alpha_i)$$

We can also use the ABBV formula to calculate the equivariant (and ordinary) characteristic numbers of the tangent bundle.

*Remark 4.3.3.* The equivariant Chern classes  $(c^T)^I$  are integral. Moreover, the canonical classes  $\tau_S$  are actually the equivariant Schubert classes, hence also integral. Therefore, the coefficients  $N, K, \bar{K}$  and characteristic numbers  $\int (c^T)^I, \int c^I$  are all integral.

## 5. EQUIVARIANT COHOMOLOGY RING OF REAL GRASSMANNIAN

In this section, we give the GKM description and Borel description of equivariant cohomology ring of real Grassmannian, together with the canonical basis and characteristic basis of the additive structure. We use the notation  $G_k(\mathbb{R}^n)$  for the Grassmannian of  $k$ -dimensional real subspaces in  $\mathbb{R}^n$ .

The dimensions of  $G_k(\mathbb{R}^n)$  is  $k(n-k)$ . Therefore,  $G_{2k}(\mathbb{R}^{2n}), G_{2k}(\mathbb{R}^{2n+1}), G_{2k+1}(\mathbb{R}^{2n+1})$  are even dimensional, but  $G_{2k+1}(\mathbb{R}^{2n+2})$  is odd dimensional. Moreover, the real Grassmannians  $G_k(\mathbb{R}^n)$  differ from each other on Poincaré series and orientability according to the parities of  $k$  and  $n$ .

**Theorem 5.0.1** (Poincaré series of real Grassmannians, (Casian&Kodama [CK13])). *The relations between Poincaré series of real Grassmannians and complex Grassmannians are given as:*

$$\begin{aligned} P_{G_{2k}(\mathbb{R}^{2n})}(t) &= P_{G_{2k}(\mathbb{R}^{2n+1})}(t) = P_{G_{2k+1}(\mathbb{R}^{2n+1})}(t) = P_{G_k(\mathbb{C}^n)}(t^2) \\ P_{G_{2k+1}(\mathbb{R}^{2n+2})}(t) &= (1 + t^{2n+1})P_{G_k(\mathbb{C}^n)}(t^2) \end{aligned}$$

*Remark 5.0.4.* The Poincaré series of complex Grassmannian is (see Bott&Tu [BT82])

$$P_{G_k(\mathbb{C}^n)}(t) = \frac{(1 - t^2) \cdots (1 - t^{2n})}{(1 - t^2) \cdots (1 - t^{2k})(1 - t^2) \cdots (1 - t^{2(n-k)})}$$

Using the relations between Poincaré series, we see that  $G_{2k}(\mathbb{R}^{2n})$  and  $G_{2k+1}(\mathbb{R}^{2n+2})$  have non-zero top Betti numbers, hence orientable; however,  $G_{2k}(\mathbb{R}^{2n+1})$  and  $G_{2k+1}(\mathbb{R}^{2n+1})$  have zero top Betti numbers, hence non-orientable.

**5.1. GKM description of real Grassmannian.** Similar to the case of complex Grassmannians, we will show the real Grassmannians also have appropriate torus actions that are equivariantly formal and GKM.

First, we specify the torus actions on real Grassmannians. Write the coordinates on  $\mathbb{R}^{2n}$  as  $(x_1, y_1, \dots, x_n, y_n)$ . Let  $T^n$  act on  $\mathbb{R}^{2n}, \mathbb{R}^{2n+1}, \mathbb{R}^{2n+2}$  so that the  $i$ -th  $S^1$ -component of  $T^n$  exactly rotates the  $i$ -th pairs of real coordinates  $(x_i, y_i)$  and leaves the remaining coordinates free, hence we can write  $\mathbb{R}^{2n} = \oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2$ ,  $\mathbb{R}^{2n+1} = (\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0$  and  $\mathbb{R}^{2n+2} = (\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0^2$  for their decompositions into weighted subspaces, where  $[\alpha_i] \in \mathbb{t}_{\mathbb{Z}}^* / \pm 1$ . These actions induce  $T^n$  actions on  $G_{2k}(\mathbb{R}^{2n}), G_{2k}(\mathbb{R}^{2n+1}), G_{2k+1}(\mathbb{R}^{2n+1})$  and  $G_{2k+1}(\mathbb{R}^{2n+2})$ .

**Proposition 5.1.1** (1-skeleton of real Grassmannian). *The fixed points, isotropy weights and 1-skeleta of real Grassmannians can be given as*

- (1) For  $G_{2k}(\mathbb{R}^{2n})$ , there are  $\binom{n}{k}$  fixed points of the form  $V_S = \oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2$ , where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . The isotropy weights at  $V_S$  are  $\{[\alpha_j \pm \alpha_i] \mid i \in S, j \notin S\}$ , among which both  $[\alpha_j + \alpha_i]$  and  $[\alpha_j - \alpha_i]$  join  $V_S$  via 2-sphere to  $V_{(S \setminus \{i\}) \cup \{j\}}$ .
- (2) For  $G_{2k}(\mathbb{R}^{2n+1})$ , there are  $\binom{n}{k}$  fixed points of the form  $V_S = \oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2$ , where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . The isotropy weights at  $V_S$  are  $\{[\alpha_j \pm \alpha_i] \mid i \in S, j \notin S\} \cup \{[\alpha_i] \mid i \in S\}$ , among which both  $[\alpha_j + \alpha_i]$  and  $[\alpha_j - \alpha_i]$  join  $V_S$  via 2-sphere to  $V_{(S \setminus \{i\}) \cup \{j\}}$ , and  $[\alpha_i]$  joins  $V_S$  via a  $\mathbb{R}P^2$  to no other fixed points.
- (3) For  $G_{2k+1}(\mathbb{R}^{2n+1})$ , there are  $\binom{n}{k}$  fixed points of the form  $V_S = (\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0$ , where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . The isotropy weights at  $V_S$  are  $\{[\alpha_j \pm \alpha_i] \mid i \in S, j \notin S\} \cup \{[\alpha_j] \mid j \notin S\}$ , among which both  $[\alpha_j + \alpha_i]$  and  $[\alpha_j - \alpha_i]$  join  $V_S$  via 2-sphere to  $V_{(S \setminus \{i\}) \cup \{j\}}$ , and  $[\alpha_j]$  joins  $V_S$  via a  $\mathbb{R}P^2$  to no other fixed points.
- (4) For  $G_{2k+1}(\mathbb{R}^{2n+2})$ , there are  $\binom{n}{k}$  fixed circles of the form  $C_S = \{(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0 \mid L_0 \in \mathbb{P}(\mathbb{R}_0^2)\} \cong \mathbb{R}P^1$ , where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . The isotropy weights at  $C_S$  are  $\{[\alpha_j \pm \alpha_i] \mid i \in S, j \notin S\} \cup \{[\alpha_i] \mid i \in S\} \cup \{[\alpha_j] \mid j \notin S\}$ , among which both  $[\alpha_j + \alpha_i]$  and  $[\alpha_j - \alpha_i]$  join  $C_S$  via a  $S^2 \times \mathbb{R}P^1$  to  $C_{(S \setminus \{i\}) \cup \{j\}}$ , and  $[\alpha_i], [\alpha_j]$  join  $C_S$  via a  $\mathbb{R}P^3$  to no other fixed circles.

The proof will be conducted in three steps on the verifications of fixed points, isotropy weights and 1-skeleta.

**5.1.1. Fixed points.** Similar to the observation in the case of complex Grassmannians, the  $T^n$ -fixed points of real Grassmannians are exactly the appropriate dimensional sub-representations of the ambient representations. The verification of sub-representations of  $\mathbb{R}^{2n} = \oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2$ ,  $\mathbb{R}^{2n+1} = (\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0$  for the real Grassmannians  $G_{2k}(\mathbb{R}^{2n}), G_{2k}(\mathbb{R}^{2n+1})$  and  $G_{2k+1}(\mathbb{R}^{2n+1})$  are straightforward. Let's focus on the  $2k+1$  dimensional sub-representations of  $\mathbb{R}^{2n+2} = (\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0^2$ . Notice that the sub-representation is odd dimensional, hence must have exactly one dimension in the part of trivial representation, therefore has the form  $(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0$  where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$  and  $L_0 \in \mathbb{P}(\mathbb{R}_0^2)$ . For each  $k$ -element subset  $S$ , the connected component  $C_S = \{(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0 \mid L_0 \in \mathbb{P}(\mathbb{R}_0^2)\} \cong \mathbb{R}P^1$  gives a fixed circle isolated from the other fixed circles. This gives all the fixed points of the  $T^n$  action on  $G_{2k+1}(\mathbb{R}^{2n+2})$ .

*Remark 5.1.1.* Because of the one-to-one correspondence between a  $k$ -element subset  $S \subset \{1, \dots, n\}$  with a fixed point or circle, sometimes we will use  $S$  directly to mean a fixed point or circle.

**5.1.2. Isotropy weights.** Fixing a  $k$ -element subset  $S$ , let's describe the tangent spaces at the fixed points in the four cases of real Grassmannians.

- (1) The tangent space at  $\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2 \in G_{2k}(\mathbb{R}^{2n})$  is

$$\text{Hom}_{\mathbb{R}}\left(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2, \oplus_{j \notin S} \mathbb{R}_{[\alpha_j]}^2\right) \cong (\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} (\oplus_{j \notin S} \mathbb{R}_{[\alpha_j]}^2) \cong \oplus_{i \in S} \oplus_{j \notin S} ((\mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} \mathbb{R}_{[\alpha_j]}^2)$$

- (2) The tangent space at  $\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2 \in G_{2k}(\mathbb{R}^{2n+1})$  is

$$\text{Hom}_{\mathbb{R}}\left(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2, (\oplus_{j \notin S} \mathbb{R}_{[\alpha_j]}^2) \oplus \mathbb{R}_0\right) \cong \left(\oplus_{i \in S} \oplus_{j \notin S} ((\mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} \mathbb{R}_{[\alpha_j]}^2)\right) \oplus \oplus_{i \in S} (\mathbb{R}_{[\alpha_i]}^2)^*$$

(3) The tangent space at  $(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0 \in G_{2k+1}(\mathbb{R}^{2n+1})$  is

$$\text{Hom}_{\mathbb{R}}\left((\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0, \oplus_{j \notin S} \mathbb{R}_{[\alpha_j]}^2\right) \cong \left(\oplus_{i \in S} \oplus_{j \notin S} ((\mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} \mathbb{R}_{[\alpha_j]}^2)\right) \oplus \oplus_{j \notin S} \mathbb{R}_{[\alpha_j]}^2$$

(4) The tangent space at  $(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0 \in G_{2k+1}(\mathbb{R}^{2n+2})$ , where  $L_0 \in \mathbb{P}(\mathbb{R}_0^2)$  has a  $L_0^\perp \in \mathbb{P}(\mathbb{R}_0^2)$  such that  $L_0 \oplus L_0^\perp \cong \mathbb{R}_0^2$ , is

$$\begin{aligned} & \text{Hom}_{\mathbb{R}}\left((\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0, (\oplus_{j \notin S} \mathbb{R}_{[\alpha_j]}^2) \oplus L_0^\perp\right) \\ & \cong \left(\oplus_{i \in S} \oplus_{j \notin S} ((\mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} \mathbb{R}_{[\alpha_j]}^2)\right) \oplus \left(\oplus_{i \in S} (\mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} L_0^\perp\right) \oplus \left(\oplus_{j \notin S} L_0^* \otimes_{\mathbb{R}} \mathbb{R}_{[\alpha_j]}^2\right) \\ & \quad \oplus \left(L_0^* \otimes_{\mathbb{R}} L_0^\perp\right) \end{aligned}$$

among which the first three terms and the fourth term give respectively the normal space and tangent space of the fixed circle  $C_S = \{(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0 \mid L_0 \in \mathbb{P}(\mathbb{R}_0^2)\}$ .

The isotropy weights are then determined by the following simple lemma:

**Lemma 5.1.1.** *The weights of the tensor product  $(\mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} \mathbb{R}_{[\alpha_j]}^2$  are  $[\alpha_j - \alpha_i], [\alpha_j + \alpha_i] \in \mathfrak{t}_{\mathbb{Z}}^* / \pm 1$ .*

*Proof.* The  $T$ -action on the dual space  $(\mathbb{R}_{[\alpha_i]}^2)^*$  is defined in an invariant way so that for  $t \in T$ ,  $l \in (\mathbb{R}_{[\alpha_i]}^2)^*$ ,  $v \in \mathbb{R}_{[\alpha_i]}^2$ , and if we denote  $\langle l, v \rangle$  as the natural pairing, we should have  $\langle t \cdot l, t \cdot v \rangle = \langle l, v \rangle$  or equivalently,  $(t \cdot l)(v) = l(t^{-1} \cdot v)$ . Notice that only the  $i$ -th and  $j$ -th  $S^1$ -component of  $T^n$  have non-trivial actions on  $(\mathbb{R}_{[\alpha_i]}^2)^*$  or  $\mathbb{R}_{[\alpha_j]}^2$ , let  $e^{\sqrt{-1}\theta_i} \in S_i^1$ ,  $e^{\sqrt{-1}\theta_j} \in S_j^1$ , and write elements of  $(\mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} \mathbb{R}_{[\alpha_j]}^2$  as  $2 \times 2$  matrices, then the  $S_i^1 \times S_j^1$  action on  $(\mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} \mathbb{R}_{[\alpha_j]}^2$  can be given as:

$$(e^{\sqrt{-1}\theta_i}, e^{\sqrt{-1}\theta_j}) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{pmatrix}$$

Consider the following new basis of  $(\mathbb{R}_{[\alpha_i]}^2)^* \otimes_{\mathbb{R}} \mathbb{R}_{[\alpha_j]}^2$

$$M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad M_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad M_4 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

We get

$$\begin{aligned} (e^{\sqrt{-1}\theta_j} \cdot M_1, e^{\sqrt{-1}\theta_j} \cdot M_2) &= (M_1, M_2) \begin{pmatrix} \cos \theta_j & -\sin \theta_j \\ \sin \theta_j & \cos \theta_j \end{pmatrix} \\ (e^{\sqrt{-1}\theta_i} \cdot M_1, e^{\sqrt{-1}\theta_i} \cdot M_2) &= (M_1, M_2) \begin{pmatrix} \cos \theta_i & \sin \theta_i \\ -\sin \theta_i & \cos \theta_i \end{pmatrix} \end{aligned}$$

In other words,  $S_i^1 \times S_j^1$  acts on  $\mathbb{R}M_1 \oplus \mathbb{R}M_2$  with weight  $[\alpha_j - \alpha_i]$ . Similarly,  $S_i^1 \times S_j^1$  acts on  $\mathbb{R}M_3 \oplus \mathbb{R}M_4$  with weight  $[\alpha_j + \alpha_i]$ .  $\square$

**5.1.3. 1-skeleta.** To begin with, let's work out the  $T^2$ -action on  $G_2(\mathbb{R}^4)$ . From the previous discussions, we know that there are two fixed points  $\mathbb{R}_{[\alpha_1]}^2, \mathbb{R}_{[\alpha_2]}^2 \in G_2(\mathbb{R}^4)$ , both have the same isotropy weights  $[\alpha_2 - \alpha_1]$  and  $[\alpha_2 + \alpha_1]$ . Let  $T_{\alpha_2 - \alpha_1}$  be the subtorus of  $T^2$  with Lie algebra annihilated by  $\alpha_2 - \alpha_1$ , i.e.  $T_{\alpha_2 - \alpha_1}$  is the diagonal  $\{(t, t) \in T^2\}$ . Similarly,  $T_{\alpha_2 + \alpha_1}$ , the subtorus with Lie algebra annihilated by  $\alpha_2 + \alpha_1$ , is the anti-diagonal  $\{(t, t^{-1}) \in T^2\}$ .

Note that there is a natural diffeomorphism  $\mathcal{F} : \mathbb{C}^2 \rightarrow \mathbb{R}^4$  by forgetting the complex structure. This induces an embedding  $\mathbb{C}P^1 \hookrightarrow G_2(\mathbb{R}^4) : L \mapsto \mathcal{F}(L)$  where  $L$  is a complex line in  $\mathbb{C}^2$  and  $\mathcal{F}(L)$  its two dimensional real image in  $\mathbb{R}^4$ . Let  $J : \mathbb{C}^2 \rightarrow \mathbb{C}^2 : (z_1, z_2) \mapsto (z_1, \bar{z}_2)$  be the diffeomorphism with conjugation on the second variable. This also induces an embedding  $\mathbb{C}P^1 \hookrightarrow G_2(\mathbb{R}^4) : L \mapsto \mathcal{F}(J(L))$ . We will denote the images of the two embeddings as  $\mathbb{C}P^1$  and  $\overline{\mathbb{C}P^1}$ .

**Lemma 5.1.2.** *The fixed-point sets of  $T_{\alpha_2-\alpha_1}$  and  $T_{\alpha_2+\alpha_1}$  in  $G_2(\mathbb{R}^4)$  are  $\mathbb{CP}^1$  and  $\overline{\mathbb{CP}^1}$  respectively, i.e. the 1-skeleton of the  $T^2$ -action on  $G_2(\mathbb{R}^4)$  is  $\mathbb{CP}^1 \cup \overline{\mathbb{CP}^1}$  glued at the two  $T^2$ -fixed points  $\mathbb{R}_{[\alpha_1]}^2, \mathbb{R}_{[\alpha_2]}^2 \in G_2(\mathbb{R}^4)$ .*

*Proof.* Let  $L_0 = \mathbb{C} \oplus 0$  and  $L_\infty = 0 \oplus \mathbb{C}$  be the two complex lines in  $\mathbb{C}^2$ , they are the two poles of both  $\mathbb{CP}^1$  and  $\overline{\mathbb{CP}^1}$ , and are exactly the two  $T^2$ -fixed points  $\mathbb{R}_{[\alpha_1]}^2, \mathbb{R}_{[\alpha_2]}^2 \in G_2(\mathbb{R}^4)$ . The diagonal circle  $T_{\alpha_2-\alpha_1} = \{(t, t) \in T^2\}$  fixes  $\mathbb{CP}^1$  because  $(t, t) \cdot [z_1, z_2] = [tz_1, tz_2] = [z_1, z_2]$  trivially, hence  $\mathbb{CP}^1$  joins  $\mathbb{R}_{[\alpha_1]}^2$  to  $\mathbb{R}_{[\alpha_2]}^2$  with weight  $[\alpha_2 - \alpha_1]$ . Similarly,  $\overline{\mathbb{CP}^1}$  joins  $\mathbb{R}_{[\alpha_1]}^2$  to  $\mathbb{R}_{[\alpha_2]}^2$  with weight  $[\alpha_2 + \alpha_1]$ . The 2-spheres  $\mathbb{CP}^1$  and  $\overline{\mathbb{CP}^1}$  exhaust all the  $T^2$ -fixed points and the isotropy weights, therefore give the 1-skeleton of the  $T^2$ -action on  $G_2(\mathbb{R}^4)$ .  $\square$

Generally, let  $T_{\alpha_j-\alpha_i}$  and  $T_{\alpha_j+\alpha_i}$  be the subtori of  $T^n$  with Lie algebras annihilated by  $\alpha_j - \alpha_i$  and  $\alpha_j + \alpha_i$  respectively. For the  $T^n$ -action on  $\mathbb{R}^{2n} = \bigoplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2$ , the fixed-point sets of  $T_{\alpha_j-\alpha_i}$  and  $T_{\alpha_j+\alpha_i}$  on  $G_2(\mathbb{R}_{[\alpha_i]}^2 \oplus \mathbb{R}_{[\alpha_j]}^2)$  are two 2-spheres sharing the poles which are exactly the two  $T^n$ -fixed points  $\mathbb{R}_{[\alpha_i]}^2, \mathbb{R}_{[\alpha_j]}^2 \in G_2(\mathbb{R}_{[\alpha_i]}^2 \oplus \mathbb{R}_{[\alpha_j]}^2)$ . We will denote the 2-spheres as  $S_{[\alpha_j-\alpha_i]}^2$  and  $S_{[\alpha_j+\alpha_i]}^2$  and keep in mind that every element  $V$  in  $S_{[\alpha_j-\alpha_i]}^2$  or  $S_{[\alpha_j+\alpha_i]}^2$  is a 2-plane in  $\mathbb{R}_{[\alpha_i]}^2 \oplus \mathbb{R}_{[\alpha_j]}^2$ .

Now we are ready to describe the 1-skeleta of the  $T^n$  actions on the four types of real Grassmannians. Let  $S$  be a  $k$ -element subset of  $\{1, 2, \dots, n\}$ , and  $i \in S, j \notin S$ .

- (1) For  $G_{2k}(\mathbb{R}^{2n})$ , the  $T^n$ -fixed point  $V_S = \bigoplus_{i' \in S} \mathbb{R}_{[\alpha_{i'}]}^2$  is joined to  $V_{(S \setminus \{i\}) \cup \{j\}}$  via  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus V \mid V \in S_{[\alpha_j-\alpha_i]}^2\} \cong S^2$  of weight  $[\alpha_j - \alpha_i]$  and also via  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus V \mid V \in S_{[\alpha_j+\alpha_i]}^2\} \cong S^2$  of weight  $[\alpha_j + \alpha_i]$ .
- (2) For  $G_{2k}(\mathbb{R}^{2n+1})$ , the  $T^n$ -fixed point  $V_S = \bigoplus_{i' \in S} \mathbb{R}_{[\alpha_{i'}]}^2$  is joined to  $V_{(S \setminus \{i\}) \cup \{j\}}$  via  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus V \mid V \in S_{[\alpha_j-\alpha_i]}^2\} \cong S^2$  of weight  $[\alpha_j - \alpha_i]$  and also via  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus V \mid V \in S_{[\alpha_j+\alpha_i]}^2\} \cong S^2$  of weight  $[\alpha_j + \alpha_i]$ . Moreover,  $V_S$  is contained in  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus V \mid V \in G_2(\mathbb{R}_{[\alpha_i]}^2 \oplus \mathbb{R}_0)\} \cong \mathbb{RP}^2$  of weight  $[\alpha_i]$  without other fixed points.
- (3) For  $G_{2k+1}(\mathbb{R}^{2n+1})$ , the  $T^n$ -fixed point  $V_S = (\bigoplus_{i' \in S} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus \mathbb{R}_0$  is joined to  $V_{(S \setminus \{i\}) \cup \{j\}}$  via  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus V \oplus \mathbb{R}_0 \mid V \in S_{[\alpha_j-\alpha_i]}^2\} \cong S^2$  of weight  $[\alpha_j - \alpha_i]$  and also via  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus V \oplus \mathbb{R}_0 \mid V \in S_{[\alpha_j+\alpha_i]}^2\} \cong S^2$  of weight  $[\alpha_j + \alpha_i]$ . Moreover,  $V_S$  is contained in  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus L \mid L \in \mathbb{P}(\mathbb{R}_{[\alpha_j]}^2 \oplus \mathbb{R}_0)\} \cong \mathbb{RP}^2$  of weight  $[\alpha_j]$  without other fixed points.
- (4) For  $G_{2k+1}(\mathbb{R}^{2n+2})$ , the  $T^n$ -fixed circle  $C_S = \{(\bigoplus_{i' \in S} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus L_0 \mid L_0 \in \mathbb{P}(\mathbb{R}_0^2)\} \cong \mathbb{RP}^1$  is joined to  $C_{(S \setminus \{i\}) \cup \{j\}}$  via  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus V \oplus L_0 \mid V \in S_{[\alpha_j-\alpha_i]}^2, L_0 \in \mathbb{P}(\mathbb{R}_0^2)\} \cong S^2 \times \mathbb{RP}^1$  with weight  $[\alpha_j - \alpha_i]$  and also via  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus V \oplus L_0 \mid V \in S_{[\alpha_j+\alpha_i]}^2, L_0 \in \mathbb{P}(\mathbb{R}_0^2)\} \cong S^2 \times \mathbb{RP}^1$  with weight  $[\alpha_j + \alpha_i]$ . Moreover,  $C_S$  is contained in  $\{(\bigoplus_{i' \in S \setminus \{i\}} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus W \mid W \in G_3(\mathbb{R}_{[\alpha_i]}^2 \oplus \mathbb{R}_0^2)\} \cong \mathbb{RP}^3$  and  $\{(\bigoplus_{i' \in S} \mathbb{R}_{[\alpha_{i'}]}^2) \oplus L \mid L \in \mathbb{P}(\mathbb{R}_{[\alpha_j]}^2 \oplus \mathbb{R}_0^2)\} \cong \mathbb{RP}^3$  of weights  $[\alpha_i]$  and  $[\alpha_j]$  respectively without other fixed points.

5.1.4. *GKM graphs of real Grassmannians.* Since  $G_{2k}(\mathbb{R}^{2n}), G_{2k}(\mathbb{R}^{2n+1}), G_{2k+1}(\mathbb{R}^{2n+1})$  are even dimensional, but  $G_{2k+1}(\mathbb{R}^{2n+2})$  is odd dimensional, we will construct GKM graphs according to the parity of dimensions.

- (1) For even dimensional Grassmannian  $G_{2k}(\mathbb{R}^{2n})$ , the GKM graph is based on the Johnson graph: a vertex  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ ; two different vertices  $S, S'$  with  $S \cup \{j\} = S' \cup \{i\}$  are joined by two edges with weights  $[\alpha_j - \alpha_i]$  and  $[\alpha_j + \alpha_i]$ .
- (2) For even dimensional Grassmannians  $G_{2k}(\mathbb{R}^{2n+1}), G_{2k+1}(\mathbb{R}^{2n+1})$ , the **effective** GKM graph by ignoring  $\mathbb{RP}^2$ 's of the 1-skeleton is the same as the GKM graph of  $G_{2k}(\mathbb{R}^{2n})$ .
- (3) For odd dimensional Grassmannian  $G_{2k+1}(\mathbb{R}^{2n+2})$ , the GKM graph consists of
  - **vertices:** For each  $k$ -element subset  $S \subseteq \{1, 2, \dots, n\}$ , there is a  $\circ$ .
  - ◻ **vertices, weights:** For each pair of different vertices  $S, S'$  with  $S \cup \{j\} = S' \cup \{i\}$ , there are two  $\square$ 's weighted  $[\alpha_j - \alpha_i]$  and  $[\alpha_j + \alpha_i]$  respectively. Near each  $S$ , there are  $n$  extra  $\square$ 's weighted  $[\alpha_1], \dots, [\alpha_n]$  respectively.

**Edges:** For each  $\circ$  with symbol  $S$ , we join it to all the nearby  $\square$  vertices of its isotropy weights.

5.1.5. *Formality, cohomology and canonical basis of real Grassmannian.* We have given the 1-skeleton and GKM graph for real Grassmannian  $G_k(\mathbb{R}^n)$  under appropriate torus actions. To apply the generalized GKM theorems in even and odd dimensions, we still need to verify that those torus actions on  $G_k(\mathbb{R}^n)$  are equivariantly formal.

**Proposition 5.1.2** (Equivariant formality of torus actions on real Grassmannians). *The total Betti numbers of  $G_k(\mathbb{R}^n)$  and of its fixed-point set are equal:*

- (1) *For the  $T^n$ -actions on  $G_{2k}(\mathbb{R}^{2n})$ ,  $G_{2k}(\mathbb{R}^{2n+1})$ ,  $G_{2k+1}(\mathbb{R}^{2n+1})$ , the isolated fixed points  $V_S$  are all parametrized by  $\mathcal{S} = \{S \subseteq \{1, 2, \dots, n\} \mid \#S = k\}$ , and we have*

$$\sum \dim H^*(G_{2k}(\mathbb{R}^{2n})) = \sum \dim H^*(G_{2k}(\mathbb{R}^{2n+1})) = \sum \dim H^*(G_{2k+1}(\mathbb{R}^{2n+1})) = \#\mathcal{S} = \binom{n}{k}$$

- (2) *For the  $T^n$ -action on  $G_{2k+1}(\mathbb{R}^{2n+2})$ , the isolated fixed circles  $C_S$  are also indexed on  $\mathcal{S} = \{S \subseteq \{1, 2, \dots, n\} \mid \#S = k\}$ , and we have*

$$\sum \dim H^*(G_{2k+1}(\mathbb{R}^{2n+1})) = \#\mathcal{S} \cdot \sum \dim H^*(S^1) = 2 \binom{n}{k}$$

Therefore, the torus actions on  $G_k(\mathbb{R}^n)$  are equivariantly formal.

*Proof.* The verification is based on the equivalence (6) of the criteria of equivariant formality. The total Betti numbers of fixed-point sets in the above both cases are clear using Theorem 5.1.1. The total Betti numbers of  $G_k(\mathbb{R}^n)$  can be calculated from the Casian-Kodama formula in Theorem 5.0.1 by substituting  $t = 1$  in the Poincaré series.

$$\begin{aligned} \sum \dim H^*(G_{2k}(\mathbb{R}^{2n})) &= \sum \dim H^*(G_{2k}(\mathbb{R}^{2n+1})) = \sum \dim H^*(G_{2k+1}(\mathbb{R}^{2n+1})) = \sum \dim H^*(G_k(\mathbb{C}^n)) \\ &\quad \sum \dim H^*(G_{2k+1}(\mathbb{R}^{2n+2})) = 2 \sum \dim H^*(G_k(\mathbb{C}^n)) \end{aligned}$$

On the other hand, by the formality of the  $T^n$ -action on  $G_k(\mathbb{C}^n)$ , which also has isolated points parametrized by  $\mathcal{S}$ , we have

$$\sum \dim H^*(G_k(\mathbb{C}^n)) = \#\mathcal{S} = \binom{n}{k}$$

Therefore, total Betti numbers of  $G_k(\mathbb{R}^n)$  and of its fixed-point set are equal, and the torus actions on  $G_k(\mathbb{R}^n)$  are equivariantly formal.  $\square$

With the verifications of GKM conditions and equivariant formality, we can give the GKM description of the torus actions on  $G_k(\mathbb{R}^n)$  by applying the generalized GKM Theorems 3.2.1 and 3.3.1 in even and odd dimensions.

**Theorem 5.1.1** (GKM description of real Grassmannians). *Let  $\mathcal{S}$  be the collection of  $k$ -element subsets of  $\{1, 2, \dots, n\}$ .*

- (1) *For even dimensional Grassmannians  $G_{2k}(\mathbb{R}^{2n})$ ,  $G_{2k}(\mathbb{R}^{2n+1})$ ,  $G_{2k+1}(\mathbb{R}^{2n+1})$  with  $T^n$ -actions, they have the same equivariant cohomology*

$$\{f : \mathcal{S} \rightarrow \mathbb{Q}[\alpha_1, \dots, \alpha_n] \mid f_S \equiv f_{S'} \pmod{\alpha_j^2 - \alpha_i^2} \text{ for } S, S' \in \mathcal{S} \text{ with } S \cup \{j\} = S' \cup \{i\}\}$$

- (2) *For odd dimensional Grassmannian  $G_{2k+1}(\mathbb{R}^{2n+2})$  with  $T^n$ -action, an element of the equivariant cohomology is a set of polynomial pairs  $(f_S, g_S\theta)$  to each  $\circ$ -vertex  $S$  where  $\theta$  is the unit volume form of  $S^1$  such that*

$$(a) \ g_S \equiv 0 \pmod{\prod_{i=1}^n \alpha_i} \text{ for every } S$$

$$(b) \ f_S \equiv f_{S'}, \quad g_S \equiv g_{S'} \pmod{\alpha_j^2 - \alpha_i^2} \text{ for } S, S' \in \mathcal{S} \text{ with } S \cup \{j\} = S' \cup \{i\}$$

*Remark 5.1.2.* For convenience, we will write an element  $f \in H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))$  as  $(f_S)_{S \in \mathcal{S}}$  and an element  $(f, g\theta) \in H_{T^n}^*(G_{2k+1}(\mathbb{R}^{2n+2}))$  as  $(f_S + g_S\theta)_{S \in \mathcal{S}}$ , which are understood as tuples indexed on  $S \in \mathcal{S}$ .

*Remark 5.1.3.* In the 1-skeleton of the odd dimensional Grassmannian  $G_{2k+1}(\mathbb{R}^{2n+2})$ , every  $\mathbb{R}P^3_{[\alpha_i]}$  containing a unique fixed circle  $C_S$  contributes a relation  $g_S \equiv 0 \pmod{\alpha_i}$ ; every  $S^2 \times \mathbb{R}P^1$  with weight  $\alpha_j \pm \alpha_i$  and two fixed circles  $C_S, C_{S'}$  contributes two relations  $f_S \equiv f_{S'}, g_S \equiv g_{S'} \pmod{\alpha_j \pm \alpha_i}$ . These simple components in 1-skeleton resolve the sign issues in odd dimensional GKM Theorem 3.3.1.

*Remark 5.1.4.* Note that in the above description, we have condensed some congruence relations because  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$  is a unique-factorization domain.

$$\begin{cases} f_S \equiv f_{S'} \pmod{\alpha_j - \alpha_i} \\ f_S \equiv f_{S'} \pmod{\alpha_j + \alpha_i} \end{cases} \iff f_S \equiv f_{S'} \pmod{\alpha_j^2 - \alpha_i^2}$$

$$\begin{cases} g_S \equiv 0 \pmod{\alpha_1} \\ \vdots \\ g_S \equiv 0 \pmod{\alpha_n} \end{cases} \iff g_S \equiv 0 \pmod{\prod_{i=1}^n \alpha_i}$$

Notice the similarity among the GKM descriptions of the even and odd dimensional real Grassmannians and the complex Grassmannian, we have

**Theorem 5.1.2** (Relations among equivariant cohomologies of real and complex Grassmannians). *The relations between the equivariant cohomologies of even, odd dimensional real Grassmannians and complex Grassmannian are*

- (1) *There are a series of  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -algebra isomorphisms:*

$$H_{T^n}^*(G_{2k}(\mathbb{R}^{2n})) \cong H_{T^n}^*(G_{2k}(\mathbb{R}^{2n+1})) \cong H_{T^n}^*(G_{2k+1}(\mathbb{R}^{2n+1}))$$

- (2) *There is an element  $r^T \in H_{T^n}^{2n+1}(G_{2k+1}(\mathbb{R}^{2n+2}))$  such that  $(r^T)^2 = 0$ , and there is a  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -algebra isomorphism*

$$H_{T^n}^*(G_{2k+1}(\mathbb{R}^{2n+2})) \cong H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))[r^T]/(r^T)^2$$

- (3) *There is a  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -algebra monomorphism:*

$$H_{T^n}^*(G_{2k}(\mathbb{R}^{2n})) \hookrightarrow H_{T^n}^*(G_k(\mathbb{C}^n))$$

*Proof.* All the Grassmannians with  $T^n$ -action are modelled on the same Johnson graph  $J(n, k)$  with slightly different congruence relations.

- (1) This has been proved in Theorem 5.1.1.  
(2) From Theorem 5.1.1, the GKM descriptions of even and odd dimensional real Grassmannians have the same congruence relations on the  $f_S$  polynomials:

$$f_S \equiv f_{S'} \pmod{\alpha_j^2 - \alpha_i^2} \quad \text{for } S, S' \in \mathcal{S} \text{ with } S \cup \{j\} = S' \cup \{i\}$$

But the odd dimensional real Grassmannian has extra part of  $g_S \theta$  with congruence relations:

- (a)  $g_S \equiv 0 \pmod{\prod_{i=1}^n \alpha_i}$  for every  $S$   
(b)  $g_S \equiv g_{S'} \pmod{\alpha_j^2 - \alpha_i^2}$  for  $S, S' \in \mathcal{S}$  with  $S \cup \{j\} = S' \cup \{i\}$

The first set of congruence relations means that

$$g_S = \left( \prod_{i=1}^n \alpha_i \right) \cdot h_S$$

for a polynomial  $h_S \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$  and for every  $S$ . Substitute into the second set of congruence relations, and note that  $\prod_{i=1}^n \alpha_i$  is coprime with  $\alpha_j^2 - \alpha_i^2$ , then we get

$$h_S \equiv h_{S'} \pmod{\alpha_j^2 - \alpha_i^2} \quad \text{for } S, S' \in \mathcal{S} \text{ with } S \cup \{j\} = S' \cup \{i\}$$

exactly the same as the congruence relations on the  $f_S$  polynomials. Denote

$$r^T = \left( \left( \prod_{i=1}^n \alpha_i \right) \theta \right)_{S \in \mathcal{S}}$$

which has  $(r^T)^2 = 0$  because  $\theta$  is the unit volume form of  $S^1$ , and has degree  $2n + 1$  because each  $\alpha_i$  is of degree 2 in cohomology. Then we can write

$$(f_S + g_S \theta)_{S \in \mathcal{S}} = (f_S)_{S \in \mathcal{S}} + r^T \cdot (h_S)_{S \in \mathcal{S}}$$

This establishes the bijection

$$H_{T^n}^*(G_{2k+1}(\mathbb{R}^{2n+2})) \cong H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))[r^T]/(r^T)^2$$

which can be easily verified to preserve additive, multiplicative and  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -module structures.

- (3) From Theorem 5.1.1, the GKM descriptions of even dimensional real Grassmannian has the congruence relations on the  $f_S$  polynomials:

$$f_S \equiv f_{S'} \pmod{\alpha_j^2 - \alpha_i^2} \quad \text{for } S, S' \in \mathcal{S} \text{ with } S \cup \{j\} = S' \cup \{i\}$$

which automatically satisfy the congruence relations on the  $f_S$  polynomials for the complex Grassmannian in Theorem 4.1.1:

$$f_S \equiv f_{S'} \pmod{\alpha_j - \alpha_i} \quad \text{for } S, S' \in \mathcal{S} \text{ with } S \cup \{j\} = S' \cup \{i\}$$

This establishes the injection

$$H_{T^n}^*(G_{2k}(\mathbb{R}^{2n})) \hookrightarrow H_{T^n}^*(G_k(\mathbb{C}^n))$$

which is also easy to verify as a  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -algebra monomorphism. □

*Remark 5.1.5.* These maps between cohomologies are actually induced from natural  $T^n$ -equivariant maps between spaces:  $G_{2k}(\mathbb{R}^{2n}) \rightarrow G_{2k}(\mathbb{R}^{2n+1})$ ,  $G_{2k}(\mathbb{R}^{2n}) \rightarrow G_{2k}(\mathbb{R}^{2n}) \times G_1(\mathbb{R}) \rightarrow G_{2k+1}(\mathbb{R}^{2n+1})$ ,  $G_{2k}(\mathbb{R}^{2n}) \rightarrow G_{2k}(\mathbb{R}^{2n}) \times G_1(\mathbb{R}^2) \rightarrow G_{2k+1}(\mathbb{R}^{2n+2})$ , and  $G_k(\mathbb{C}^n) \rightarrow G_{2k}(\mathbb{R}^{2n})$ . This claim will be clear when we understand more about either canonical basis or characteristic basis of the equivariant cohomology in the following discussions.

*Remark 5.1.6.* The  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -algebra isomorphisms in Theorem 5.1.2 give ring isomorphisms among ordinary cohomologies of real Grassmannians. But the ordinary version  $H^*(G_{2k}(\mathbb{R}^{2n})) \hookrightarrow H^*(G_k(\mathbb{C}^n))$  is not injective simply due to fact that  $G_{2k}(\mathbb{R}^{2n})$  is of dimension  $4k(n - k)$ , twice the real dimension of  $G_k(\mathbb{C}^n)$ .

**Theorem 5.1.3** (Canonical basis of even dimensional real Grassmannians). *There is a self-indexing Morse function on  $\mathcal{S}$*

$$\psi : \mathcal{S} \longrightarrow \mathbb{R} : S \longmapsto 4 \left( \sum_{i \in S} i \right) - 2k(k + 1)$$

and a canonical class  $\sigma_S \in H_{T^n}^{\psi(S)}(G_{2k}(\mathbb{R}^{2n}), \mathbb{Q})$  for each  $S \in \mathcal{S}$  such that

- (1)  $\sigma_S$  is supported upward, i.e.  $\sigma_S(S') = 0$  if  $\psi(S') \leq \psi(S)$
- (2)  $\sigma_S(S) = \prod' (\alpha_j^2 - \alpha_i^2)$  where the product is taken over the weights at  $S$  connecting to  $S'$  with  $\psi(S') < \psi(S)$

Moreover,  $\{\sigma_S, S \in \mathcal{S}\}$  give an additive  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -basis of  $H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}), \mathbb{Q})$ .

*Proof.* By Theorem 5.1.2, we can identify  $H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))$  with its embedded image in  $H_{T^n}^*(G_k(\mathbb{C}^n))$ . Recall from Theorem 4.1.2 on the canonical classes of complex Grassmannian  $G_k(\mathbb{C}^n)$ , we used the function  $\phi = \frac{\psi}{2}$ , hence both  $\psi$  and  $\phi$  define the same partial order on  $\mathcal{S}$ . Moreover, there is a basis  $\tau_S$  of  $H_{T^n}^*(G_k(\mathbb{C}^n))$ , such that

- (1)  $\tau_S$  is supported upward, i.e.  $\tau_S(S') = 0$  if  $\phi(S') \leq \phi(S)$
- (2)  $\tau_S(S) = \prod' (\alpha_j - \alpha_i)$  where the product is taken over the weights at  $S$  connecting to  $S'$  with  $\phi(S') < \phi(S)$

Let's introduce the ring homomorphism:

$$Sq : \mathbb{Q}[\alpha_1, \dots, \alpha_n] \rightarrow \mathbb{Q}[\alpha_1, \dots, \alpha_n] : f(\alpha_1, \dots, \alpha_n) \mapsto f(\alpha_1^2, \dots, \alpha_n^2)$$

If  $f_S \equiv f_{S'} \pmod{\alpha_j - \alpha_i}$ , i.e.  $f_S - f_{S'}$  is a multiple of  $\alpha_j - \alpha_i$ , then  $Sq(f_S) - Sq(f_{S'}) = Sq(f_S - f_{S'})$  is a multiple of  $\alpha_j^2 - \alpha_i^2$ , i.e.  $Sq(f_S) \equiv Sq(f_{S'}) \pmod{\alpha_j^2 - \alpha_i^2}$ . The homomorphism  $Sq$  not only refines the congruence relations of  $H_{T^n}^*(G_k(\mathbb{C}^n))$ , but also has image in  $H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))$ , i.e.  $Sq(H_{T^n}^*(G_k(\mathbb{C}^n))) \subseteq H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))$ . Now we can define  $\sigma_S = Sq(\tau_S) \in H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))$ , and we see that this collection of classes satisfies the required properties of being supported upward and  $\sigma_S(S) = \prod' (\alpha_j^2 - \alpha_i^2)$  over weights at  $S$  connecting to  $S'$  with  $\psi(S') < \psi(S)$ .

According to Guillemin&Zara [GZ03],  $\sigma_S$  gives an additive basis of  $H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))$ .  $\square$

Since we have proved  $H_{T^n}^*(G_{2k+1}(\mathbb{R}^{2n+2})) \cong H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))[r^T]/(r^T)^2$  in Theorem 5.1.2, then

**Corollary 5.1.1** (Canonical basis of odd dimensional real Grassmannian).  $\sigma_S$  and  $r^T \sigma_S$  give an additive  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -basis of  $H_{T^n}^*(G_{2k+1}(\mathbb{R}^{2n+2}))$ .

*Remark 5.1.7.* In the case of complex Grassmannian  $G_k(\mathbb{C}^n)$ , a subset  $S \subseteq \{1, 2, \dots, n\}$  with elements  $i_1 < i_2 < \dots < i_k$  corresponds to Schubert symbol  $(i_1 - 1, i_2 - 2, \dots, i_k - k)$ ; there could be correspondences for real Grassmannians

- (1) For even dimensional Grassmannians  $G_{2k}(\mathbb{R}^{2n}), G_{2k}(\mathbb{R}^{2n+1})$ , let  $S$  consist of  $i_1 < i_2 < \dots < i_k$ , then the  $T^n$ -fixed point  $\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2$ , with pivot positions  $(2i_1 - 1, 2i_1, 2i_2 - 1, 2i_2, \dots, 2i_k - 1, 2i_k)$  in its reduced echelon form, will correspond to Schubert symbol  $(2i_1 - 2, 2i_1 - 2, 2i_2 - 4, 2i_2 - 4, \dots, 2i_k - 2k, 2i_k - 2k)$ .
- (2) For even dimensional Grassmannian  $G_{2k+1}(\mathbb{R}^{2n+1})$ , the  $T^n$ -fixed point  $\mathbb{R}_0 \oplus (\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2)$ , with pivot positions  $(1, 2i_1, 2i_1 + 1, 2i_2, 2i_2 + 1, \dots, 2i_k, 2i_k + 1)$  in its reduced echelon form, will also correspond to Schubert symbol  $(2i_1 - 2, 2i_1 - 2, 2i_2 - 4, 2i_2 - 4, \dots, 2i_k - 2k, 2i_k - 2k)$ .
- (3) For odd dimensional Grassmannian  $G_{2k+1}(\mathbb{R}^{2n+2})$ , besides the above Schubert symbols  $(2i_1 - 2, 2i_1 - 2, 2i_2 - 4, 2i_2 - 4, \dots, 2i_k - 2k, 2i_k - 2k)$ , there is the class  $r^T$ , which is conjectured by Casian&Kodama [CK13] to be the Schubert class with the hook Young diagram  $1^{2k} \times (2(n-k)+1)$  of symbol  $(1, \dots, 1, 2(n-k)+1)$  where there are  $2k$  copies of 1. Following this conjecture, we can guess that a class  $r^T \sigma_S$  with  $S$  given by  $i_1 < i_2 < \dots < i_k$ , corresponds to the Schubert symbol  $(2i_1 - 1, 2i_1 - 1, 2i_2 - 3, 2i_2 - 3, \dots, 2i_k - 2k + 1, 2i_k - 2k + 1, 2(n-k) + 1)$ .

*Remark 5.1.8.* Recall from Subsection 4.3 of the equivariant Littlewood-Richardson rule for complex Grassmannian  $G_k(\mathbb{C}^n)$

$$\tau_S \tau_{S'} = \sum_{S''} N_{S, S'}^{S''} \tau_{S''}$$

where  $N_{S, S'}^{S''} \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$ . If we apply the ring homomorphism  $Sq$  on both sides, then we get the equivariant Littlewood-Richardson rule for real Grassmannian  $G_{2k}(\mathbb{R}^{2n})$

$$\sigma_S \sigma_{S'} = \sum_{S''} Sq(N_{S, S'}^{S''}) \sigma_{S''}$$

where  $Sq(N_{S, S'}^{S''}) \in \mathbb{Q}[\alpha_1^2, \dots, \alpha_n^2]$  is obtained from  $N_{S, S'}^{S''}$  by replacing  $\alpha_i$  to be  $\alpha_i^2$ . Since  $Sq$  keeps constant term unchanged, the Littlewood-Richardson rules for ordinary cohomology of complex Grassmannian  $G_k(\mathbb{C}^n)$  and real Grassmannian  $G_{2k}(\mathbb{R}^{2n})$  are the same.

**5.2. Borel description of real Grassmannian.** Similar to Borel description of equivariant (ordinary) cohomology of complex Grassmannian using equivariant (ordinary) Chern classes, we will show there is a Borel description of equivariant (ordinary) cohomology of real Grassmannian using equivariant (ordinary) Pontryagin classes.



5.2.1. *Equivariant Pontryagin classes.* The  $T^n$  actions on  $\mathbb{R}^{2n} = \oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2$ ,  $\mathbb{R}^{2n+1} = (\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0$  and  $\mathbb{R}^{2n+2} = (\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0^2$  induce actions on  $G_{2k}(\mathbb{R}^{2n})$ ,  $G_{2k}(\mathbb{R}^{2n+1})$ ,  $G_{2k+1}(\mathbb{R}^{2n+1})$  and  $G_{2k+1}(\mathbb{R}^{2n+2})$ . These actions induce further actions on the canonical bundles  $\gamma$  and complementary bundles  $\bar{\gamma}$  over those Grassmannians. Then we can consider their equivariant Pontryagin classes  $p^T = p^T(\gamma)$  and  $\bar{p}^T = p^T(\bar{\gamma})$ .

First, let's compute a warm-up example of equivariant Pontryagin classes.

**Lemma 5.2.1.** *The total equivariant Pontryagin class of the vector space  $\mathbb{R}_{[\alpha]}^2$  with weight  $[\alpha] \in \mathfrak{t}_{\mathbb{Z}}^* / \pm 1$  over a point is  $1 + \alpha^2$ .*

*Proof.* Think of the elements of  $\mathbb{R}_{[\alpha]}^2$  as  $2 \times 1$  column vectors. For a Lie algebra element  $\xi \in \mathfrak{t}$ , the action of its group element  $\exp(\xi) \in T$  on  $\mathbb{R}_{[\alpha]}^2$  is given as a  $2 \times 2$  real matrix

$$\begin{pmatrix} \cos(\alpha(\xi)) & -\sin(\alpha(\xi)) \\ \sin(\alpha(\xi)) & \cos(\alpha(\xi)) \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \cos(\alpha(\xi)) & \sin(\alpha(\xi)) \\ -\sin(\alpha(\xi)) & \cos(\alpha(\xi)) \end{pmatrix}$$

Tensoring  $\mathbb{R}_{[\alpha]}^2$  over  $\mathbb{R}$ -coefficients with  $\mathbb{C}$  means that we can treat the above real matrices as complex matrices. Since both of them have the same characteristic function  $\lambda^2 - 2\cos(\alpha(\xi))\lambda + 1 = (\lambda - e^{\sqrt{-1}\alpha(\xi)})(\lambda - e^{-\sqrt{-1}\alpha(\xi)})$ , the two real matrices have the same diagonalization over  $\mathbb{C}$ -coefficients:

$$\begin{pmatrix} e^{\sqrt{-1}\alpha(\xi)} & 0 \\ 0 & e^{-\sqrt{-1}\alpha(\xi)} \end{pmatrix}$$

i.e. the  $T$ -action on the complex vector space  $\mathbb{R}_{[\alpha]}^2 \otimes_{\mathbb{R}} \mathbb{C}$  has weights  $\alpha$  and  $-\alpha$ . Therefore,  $c^T(\mathbb{R}_{[\alpha]}^2 \otimes_{\mathbb{R}} \mathbb{C}) = (1 - \alpha)(1 + \alpha) = 1 - \alpha^2$ . Following Milnor-Stasheff's convention of signs, we get  $p^T(\mathbb{R}_{[\alpha]}^2) = 1 + \alpha^2$ .  $\square$

Second, let's specify the equivariant Pontryagin classes of canonical bundles, complementary bundles and tangent bundles of real Grassmannians in GKM description at each fixed point or circle of the real Grassmannians.

**Proposition 5.2.1.** *For all the four real Grassmannians  $G_{2k}(\mathbb{R}^{2n})$ ,  $G_{2k}(\mathbb{R}^{2n+1})$ ,  $G_{2k+1}(\mathbb{R}^{2n+1})$  and  $G_{2k+1}(\mathbb{R}^{2n+2})$  with  $T^n$ -actions, the equivariant Pontryagin classes  $p^T = p^T(\gamma)$  and  $\bar{p}^T = p^T(\bar{\gamma})$  of the canonical bundle and complementary bundle localized at each fixed point or circle indexed as a  $k$ -element subset  $S \in \{1, \dots, n\}$  are*

$$p^T|_S = p^T(\gamma|_S) = \prod_{i \in S} (1 + \alpha_i^2)$$

$$\bar{p}^T|_S = p^T(\bar{\gamma}|_S) = \prod_{j \notin S} (1 + \alpha_j^2)$$

with the relation  $p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2)$ . The equivariant Pontryagin classes of the tangent bundles are given at each fixed point or circle  $S$  as

$$p^T(TG_{2k}(\mathbb{R}^{2n}))|_S = \prod_{i \in S} \prod_{j \notin S} [(1 + (\alpha_j - \alpha_i)^2)(1 + (\alpha_j + \alpha_i)^2)]$$

$$p^T(TG_{2k}(\mathbb{R}^{2n+1}))|_S = \prod_{i \in S} \prod_{j \notin S} [(1 + (\alpha_j - \alpha_i)^2)(1 + (\alpha_j + \alpha_i)^2)] \prod_{i \in S} (1 + \alpha_i^2)$$

$$p^T(TG_{2k+1}(\mathbb{R}^{2n+1}))|_S = \prod_{i \in S} \prod_{j \notin S} [(1 + (\alpha_j - \alpha_i)^2)(1 + (\alpha_j + \alpha_i)^2)] \prod_{j \notin S} (1 + \alpha_j^2)$$

$$p^T(TG_{2k+1}(\mathbb{R}^{2n+2}))|_S = \prod_{i \in S} \prod_{j \notin S} [(1 + (\alpha_j - \alpha_i)^2)(1 + (\alpha_j + \alpha_i)^2)] \prod_{i \in S} (1 + \alpha_i^2) \prod_{j \notin S} (1 + \alpha_j^2)$$

*Proof.* For  $G_{2k}(\mathbb{R}^{2n})$ , at each fixed point  $S$ , we have  $\gamma|_S = \oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2$  and  $\bar{\gamma}|_S = \oplus_{j \notin S} \mathbb{R}_{[\alpha_j]}^2$ , and furthermore  $\gamma|_S \oplus \bar{\gamma}|_S = \oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2$ . The claimed expressions of the localized Pontryagin classes then follow from the Lemma

5.2.1. The cases of  $G_{2k}(\mathbb{R}^{2n+1})$ ,  $G_{2k+1}(\mathbb{R}^{2n+1})$  and  $G_{2k+1}(\mathbb{R}^{2n+2})$  are similar. For the Pontryagin classes of the tangent bundles localized at each fixed point or circle, we can apply Lemma 5.2.1 to the weight decompositions (see Subsubsection 5.1.2) of tangent bundles at each fixed point or circle.  $\square$

5.2.2. *Characteristic basis of real Grassmannians.* Think of  $p^T$  and  $\bar{p}^T$  as elements of the embedded image of  $H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))$  in  $H_{T^n}^*(G_k(\mathbb{C}^n))$  using GKM description on the Johnson graph  $J(n, k)$ . If we compare the localized expressions of  $p^T$  and  $\bar{p}^T$  with  $c^T$  and  $\bar{c}^T$  in Subsection 4.3, we get the formula

$$p^T = Sq(c^T) \quad \bar{p}^T = Sq(\bar{c}^T)$$

where the homomorphism  $Sq$  is defined in Theorem 5.1.3.

Recall in Subsection 4.3, we discussed the transformations  $K, \bar{K}$  between the characteristic monomials  $(c^T)^I = (c_1^T)^{i_1} \cdots (c_k^T)^{i_k}$  in Borel description and the canonical classes  $\tau_S$  in GKM description:

$$(c^T)^I = \sum_S K_S^I \tau_S$$

$$\tau_S = \sum_S \bar{K}_I^S (c^T)^I$$

where  $K_S^I, \bar{K}_I^S \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$ . Apply the homomorphism  $Sq$  and recall  $\sigma_S = Sq(\tau_S)$  from Theorem 5.1.3, then

$$(p^T)^I = \sum_S Sq(K_S^I) \sigma_S$$

$$\sigma_S = \sum_S Sq(\bar{K}_I^S) (p^T)^I$$

where  $Sq(K_S^I), Sq(\bar{K}_I^S) \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$ .

Since  $\sigma_S$  gives a basis of  $H_{T^n}^*(G_{2k}(\mathbb{R}^{2n}))$ , the above transformations imply:

**Theorem 5.2.1** (Equivariant characteristic basis of real Grassmannians). *The set of monomials  $(p_1^T)^{r_1} (p_2^T)^{r_2} \cdots (p_k^T)^{r_k}$  satisfying the condition  $\sum_{i=1}^k r_i \leq n-k$  forms an additive  $H_T^*(pt)$ -basis for  $H_{T^n}^*(G_{2k}(\mathbb{R}^{2n})) \cong H_{T^n}^*(G_{2k}(\mathbb{R}^{2n+1})) \cong H_{T^n}^*(G_{2k+1}(\mathbb{R}^{2n+1}))$ . Together with the set of monomials  $r^T \cdot (p_1^T)^{r_1} (p_2^T)^{r_2} \cdots (p_k^T)^{r_k}$ , they form an additive  $H_T^*(pt)$ -basis for  $H_{T^n}^*(G_{2k+1}(\mathbb{R}^{2n+2}))$ .*

Now we can give the Borel description for equivariant cohomology of real Grassmannians.

**Theorem 5.2.2** (Equivariant Borel description of even dimensional real Grassmannian). *For the even dimensional real Grassmannians  $G_{2k}(\mathbb{R}^{2n})$ ,  $G_{2k}(\mathbb{R}^{2n+1})$  and  $G_{2k+1}(\mathbb{R}^{2n+1})$  with  $T^n$ -actions, their equivariant cohomologies are the same:*

$$H_T^*(G_{2k}(\mathbb{R}^{2n}), \mathbb{Q}) \cong \frac{\mathbb{Q}[\alpha_1, \alpha_2, \dots, \alpha_n][p_1^T, p_2^T, \dots, p_k^T; \bar{p}_1^T, \bar{p}_2^T, \dots, \bar{p}_{n-k}^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2)}$$

*Proof.* For convenience, we will use short hand notations  $\mathbb{Q}[\alpha]$ ,  $\mathbb{Q}[\alpha; p^T; \bar{p}^T]$  for the polynomial rings generated on  $\alpha_i$  and  $\alpha_i, p_j^T, \bar{p}_l^T$  respectively.

If we associate to  $\alpha_i$  and the abstract symbols  $p_j^T, \bar{p}_j^T$  with gradings 2, 4j respectively, and notice that the relations in  $p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2)$  preserve gradings, then the abstract quotient ring

$$\frac{\mathbb{Q}[\alpha; p^T; \bar{p}^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2)}$$

is a graded algebra, with a  $\mathbb{Q}[\alpha]$ -algebra homomorphism:

$$\frac{\mathbb{Q}[\alpha; p^T; \bar{p}^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2)} \longrightarrow H_T^*(G_{2k}(\mathbb{R}^{2n}), \mathbb{Q})$$

by sending the abstract generators  $p_i^T, \bar{p}_j^T$  to  $p_i^T(\gamma), p_j^T(\bar{\gamma}) \in H_T^*(G_{2k}(\mathbb{R}^{2n}), \mathbb{Q})$  and extending  $\mathbb{Q}[\alpha]$ -multiplicatively to the whole ring. This is actually surjective, because  $H_T^*(G_{2k}(\mathbb{R}^{2n}), \mathbb{Q})$  has basis in terms of monomials  $(p^T(\gamma))^r$  by Theorem 5.2.1.

On the other hand,  $H_T^*(G_{2k}(\mathbb{R}^{2n}), \mathbb{Q})$  is a free  $\mathbb{Q}[\alpha]$ -module, we can construct a  $\mathbb{Q}[\alpha]$ -module homomorphism:

$$H_T^*(G_{2k}(\mathbb{R}^{2n}), \mathbb{Q}) \longrightarrow \frac{\mathbb{Q}[\alpha; p^T; \bar{p}^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2)}$$

by sending the additive basis  $(p_1^T(\gamma))^{r_1} (p_2^T(\gamma))^{r_2} \cdots (p_k^T(\gamma))^{r_k}$ , which satisfies appropriate constraints on  $r_i$ , to  $(p_1^T)^{r_1} (p_2^T)^{r_2} \cdots (p_k^T)^{r_k}$ . According to the next Lemma, this homomorphism is also surjective.

Since both of the two homomorphisms in opposite directions preserve gradings, hence at each grading are surjective between finite dimensional  $\mathbb{Q}$ -vector spaces, therefore are bijective.  $\square$

**Lemma 5.2.2.** *Every element of the abstract polynomial algebra*

$$\frac{\mathbb{Q}[\alpha_1, \dots, \alpha_n][p_1^T, \dots, p_k^T; \bar{p}_1^T, \dots, \bar{p}_{n-k}^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2)}$$

can be written as a  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -linear combination of the monomials  $(p_1)^{r_1} (p_2)^{r_2} \cdots (p_k)^{r_k}$ , where  $\sum_{i=1}^k r_i \leq n - k$ .

*Proof.* Define the ring homomorphism between two abstract polynomial algebras:

$$SQ : \mathbb{Q}[\alpha_1, \dots, \alpha_n; c_1^T, \dots, c_k^T; \bar{c}_1^T, \dots, \bar{c}_{n-k}^T] \longrightarrow \mathbb{Q}[\alpha_1, \dots, \alpha_n; p_1^T, \dots, p_k^T; \bar{p}_1^T, \dots, \bar{p}_{n-k}^T]$$

by sending the generators  $\alpha_i, c_j^T, \bar{c}_l^T$  to  $\alpha_i^2, p_j^T, \bar{p}_l^T$  and extending  $\mathbb{Q}$ -multiplicatively.

Consider the set of multi-indices  $R = \{(r_1, \dots, r_k) \mid r_i \geq 0, \sum_{i=1}^k r_i \leq n - k\}$ , and the  $n - 1$  polynomials  $f_2(\alpha, c^T, \bar{c}^T), \dots, f_n(\alpha, c^T, \bar{c}^T)$  from the gradings  $2, 4, \dots, 2n$  of the relation  $c^T \bar{c}^T = \prod_{i=1}^n (1 + \alpha_i)$ . The fact that  $(c^T)^r = (c_1^T)^{r_1} \cdots (c_k^T)^{r_k}, r \in R$  forms an additive  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -basis of  $\mathbb{Q}[\alpha; c^T; \bar{c}^T]/c^T \bar{c}^T = \prod_{i=1}^n (1 + \alpha_i)$  means that for any non-negative multi-index  $(I; J) = (I_1, \dots, I_k; J_1, \dots, J_{n-k})$ ,

$$(c^T)^I (\bar{c}^T)^J = \sum_{r \in R} A_r \cdot (c^T)^r + \sum_{s=2}^n B_s \cdot f_s$$

in  $\mathbb{Q}[\alpha; c^T; \bar{c}^T]$  for some polynomials  $A_r \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$  and  $B_s \in \mathbb{Q}[\alpha_1, \dots, \alpha_n; c_1^T, \dots, c_k^T; \bar{c}_1^T, \dots, \bar{c}_{n-k}^T]$ . Now if we apply the  $SQ$  on both sides of the above equality, we get

$$(p^T)^I (\bar{p}^T)^J = \sum_{r \in R} SQ(A_r) \cdot (p^T)^r + \sum_{s=2}^n SQ(B_s) \cdot SQ(f_s)$$

in  $\mathbb{Q}[\alpha; p^T; \bar{p}^T]$  where  $SQ(A_r) \in \mathbb{Q}[\alpha_1^2, \dots, \alpha_n^2]$  and  $SQ(B_s), SQ(f_s) \in \mathbb{Q}[\alpha_1^2, \dots, \alpha_n^2; p_1^T, \dots, p_k^T; \bar{p}_1^T, \dots, \bar{p}_{n-k}^T]$ . Notice that the  $n - 1$  polynomials  $SQ(f_2), \dots, SQ(f_n)$  exactly come from the gradings  $4, 8, \dots, 4n$  of the relation  $p^T \bar{p}^T = SQ(c^T \bar{c}^T) = SQ(\prod_{i=1}^n (1 + \alpha_i)) = \prod_{i=1}^n (1 + \alpha_i^2)$ . Therefore, the  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -linear combinations of  $(p^T)^r, r \in R$  span the quotient ring  $\mathbb{Q}[\alpha; p^T; \bar{p}^T]/p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i)$ .  $\square$

**Corollary 5.2.1** (Equivariant Borel description of odd dimensional real Grassmannian). *For the odd dimensional real Grassmannian  $G_{2k+1}(\mathbb{R}^{2n+2})$  with  $T^n$ -actions, the equivariant cohomology is:*

$$H_T^*(G_{2k+1}(\mathbb{R}^{2n+2}), \mathbb{Q}) \cong \frac{\mathbb{Q}[\alpha_1, \alpha_2, \dots, \alpha_n][p_1^T, p_2^T, \dots, p_k^T; \bar{p}_1^T, \bar{p}_2^T, \dots, \bar{p}_{n-k}^T; r^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2), (r^T)^2 = 0}$$

For a  $T^n$ -equivariantly formal space  $M$ , we can recover the ordinary cohomology from the equivariant cohomology by  $H^*(M, \mathbb{Q}) = H_T^*(M, \mathbb{Q}) \otimes_{\mathbb{Q}[\alpha_1, \dots, \alpha_n]} \mathbb{Q}$  where  $\mathbb{Q}$  has a  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -algebra structure from the constant-term morphism  $\mathbb{Q}[\alpha_1, \dots, \alpha_n] \rightarrow \mathbb{Q} : f(\alpha_1, \dots, \alpha_n) \mapsto f(0)$ . Therefore, both of Theorem 5.2.1 and Theorem 5.2.2 have ordinary versions by ignoring the  $\alpha_i$ .

Let  $r \in H^{2n+1}(G_{2k+1}(\mathbb{R}^{2n+2}), \mathbb{Q})$  be the ordinary image of the  $\tilde{r} \in H_T^{2n+1}(G_{2k+1}(\mathbb{R}^{2n+2}), \mathbb{Q})$ .

**Corollary 5.2.2** (Ordinary characteristic basis of real Grassmannians). *The set of monomials  $(p_1)^{r_1}(p_2)^{r_2} \cdots (p_k)^{r_k}$  satisfying the condition  $\sum_{i=1}^k r_i \leq n - k$  forms an additive basis for  $H^*(G_{2k}(\mathbb{R}^{2n})) \cong H^*(G_{2k}(\mathbb{R}^{2n+1})) \cong H^*(G_{2k+1}(\mathbb{R}^{2n+1}))$ . Together with the set of monomials  $r \cdot (p_1)^{r_1}(p_2)^{r_2} \cdots (p_k)^{r_k}$ , they form an additive basis for  $H^*(G_{2k+1}(\mathbb{R}^{2n+2}))$ .*

**Corollary 5.2.3** (Ordinary Borel description of real Grassmannians). *For the even dimensional real Grassmannians  $G_{2k}(\mathbb{R}^{2n})$ ,  $G_{2k}(\mathbb{R}^{2n+1})$  and  $G_{2k+1}(\mathbb{R}^{2n+1})$ , their cohomologies are the same:*

$$\frac{\mathbb{Q}[p_1, p_2, \dots, p_k; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{n-k}]}{p\bar{p} = 1}$$

*For the odd dimensional real Grassmannian  $G_{2k+1}(\mathbb{R}^{2n+2})$ , the cohomology is:*

$$\frac{\mathbb{Q}[p_1, p_2, \dots, p_k; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{n-k}; r]}{p\bar{p} = 1, r^2 = 0}$$

*where  $r \in H^{2n+1}(G_{2k+2}(\mathbb{R}^{2n+2}), \mathbb{Q})$  is the ordinary image of the  $r^T \in H_T^{2n+1}(G_{2k+2}(\mathbb{R}^{2n+2}), \mathbb{Q})$ .*

*Remark 5.2.1.* This corollary verifies Casian&Kodama's conjecture [CK13] on the ring structure of the cohomology of real Grassmannians.

*Remark 5.2.2.* For  $n \leq 7$ , the ordinary cohomology of  $G_k(\mathbb{R}^n)$  in  $\mathbb{Z}$  coefficients was computed by Junkind [Ju79].

## 6. EQUIVARIANT COHOMOLOGY RING OF ORIENTED GRASSMANNIAN

In this section, we give the GKM description and Borel description of equivariant cohomology ring of oriented Grassmannian, together with the characteristic basis of the additive structure. We use the notation  $\tilde{G}_k(\mathbb{R}^n)$  for the Grassmannian of  $k$ -dimensional oriented subspaces in  $\mathbb{R}^n$ .

The Plücker embedding of oriented Grassmannian can be given as follows: for  $V \in \tilde{G}_k(\mathbb{R}^n)$ , we can choose an ordered orthonormal basis  $v_1, \dots, v_k$  of  $V$ , then the well-defined wedge product  $v_1 \wedge \cdots \wedge v_k \in \tilde{G}_1(\wedge^k \mathbb{R}^n) = S(\wedge^k \mathbb{R}^n)$  in the unit sphere of  $\wedge^k \mathbb{R}^n$  gives the embedding  $\tilde{G}_k(\mathbb{R}^n) \hookrightarrow S(\wedge^k \mathbb{R}^n)$ .

Similar to the case of real Grassmannian, we can consider the  $T^n$ -action on  $\mathbb{R}^{2n} = \oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2$ ,  $\mathbb{R}^{2n+1} = (\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0$  and  $\mathbb{R}^{2n+2} = (\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0^2$  for their decompositions into weighted subspaces, where  $\alpha_1, \dots, \alpha_n$  are the standard basis of  $\mathfrak{t}_{\mathbb{Z}}^*$ . These actions induce  $T^n$  actions on  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ ,  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ ,  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$  and  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ . More specifically, each  $t \in T$  maps  $v_1 \wedge \cdots \wedge v_l$  where  $l = 2k, 2k+1$ , to  $t \cdot v_1 \wedge \cdots \wedge t \cdot v_l$ , and it is easy to check the map is independent from the choice of a positive orthonormal basis  $v_1, \dots, v_k$ .

Alternatively, we can think of oriented Grassmannian  $\tilde{G}_k(\mathbb{R}^n)$  as homogeneous spaces  $SO(n)/SO(k) \times SO(n-k)$ . Let  $T$  be the maximal torus of  $SO(k) \times SO(n-k)$  which has rank  $[\frac{k}{2}] + [\frac{n-k}{2}]$ , then  $T$  acts on  $SO(n)/SO(k) \times SO(n-k)$  from the left.

**6.1. Oriented Grassmannian as 2-cover over real Grassmannian.** There is a natural 2-covering of oriented Grassmannian over real Grassmannian  $\pi : \tilde{G}_k(\mathbb{R}^n) \rightarrow G_k(\mathbb{R}^n) : v_1 \wedge \cdots \wedge v_k \mapsto \text{Span}_{\mathbb{R}}(v_1, \dots, v_k)$  which induces a pull-back morphism  $\pi^* : H^*(G_k(\mathbb{R}^n)) \rightarrow H^*(\tilde{G}_k(\mathbb{R}^n))$  between their cohomologies. The non-trivial deck transformation is defined by reversing orientations  $\rho : \tilde{G}_k(\mathbb{R}^n) \rightarrow \tilde{G}_k(\mathbb{R}^n) : v_1 \wedge \cdots \wedge v_k \mapsto -(v_1 \wedge \cdots \wedge v_k)$  which induces an isomorphism  $\rho^* : H^*(\tilde{G}_k(\mathbb{R}^n)) \rightarrow H^*(\tilde{G}_k(\mathbb{R}^n))$ . Both  $\pi$  and  $\rho$  commutes with the  $T$ -actions that we introduced on the oriented Grassmannian and real Grassmannian.

For covering maps between compact spaces, or equivalently for free actions of finite groups, there is a well-known fact relating their cohomologies in rational coefficients:

**Lemma 6.1.1.** *Let  $\pi : X \rightarrow Y$  be a covering between compact topological spaces with a finite deck transformation group  $G$  which also acts on the cohomology  $H^*(X, \mathbb{Q})$ . Then  $\pi^* : H^*(Y, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q})$  is injective with image*

$H^*(X, \mathbb{Q})^G$ . This conclusion is also true for equivariant cohomology if a torus  $T$  acts on  $X$  and commutes with the action of  $G$ .

*Proof.* For a cocycle  $c$  of  $X$ , the averaged cocycle  $\frac{1}{|G|} \sum_{g \in G} gc$  is invariant under  $G$ -action, hence comes from a cocycle of  $Y$ . Consider the averaging map  $\pi_* : H^*(X, \mathbb{Q}) \rightarrow H^*(Y, \mathbb{Q}) : [c] \mapsto \frac{1}{|G|} [\sum_{g \in G} gc]$ , then the composition  $\pi_* \pi^*$  is the identity map on  $H^*(Y, \mathbb{Q})$ , hence  $\pi^*$  is injective. Note that every cohomology class in  $H^*(X, \mathbb{Q})^G$  can be represented by a  $G$ -invariant cocycle using the averaging method. This proves the image of  $\pi^*$  is exactly  $H^*(X, \mathbb{Q})^G$ .

For the  $T^n$ -equivariant version, though the Borel construction  $X \times_{T^n} (S^\infty)^n, Y \times_{T^n} (S^\infty)^n$  is not compact, we can apply the ordinary version of current Lemma to the compact approximations  $X \times_{T^n} (S^N)^n, Y \times_{T^n} (S^N)^n$  for  $N \rightarrow \infty$ .  $\square$

*Remark 6.1.1.* For the averaging method to work, we can relax the  $\mathbb{Q}$  coefficients to be any coefficient ring that contains  $\frac{1}{|G|}$ . In  $\mathbb{R}$  coefficients, the ordinary and equivariant de Rham theory together with the averaging method gives a proof without using compact approximations.

Applying this Lemma to the oriented Grassmannian as a  $T$ -equivariant 2-cover over real Grassmannian, we get

**Proposition 6.1.1.** *The pull-back morphisms of ordinary and equivariant cohomologies*

$$\pi^* : H^*(G_k(\mathbb{R}^n)) \hookrightarrow H^*(\tilde{G}_k(\mathbb{R}^n)) \quad \text{and} \quad H_T^*(G_k(\mathbb{R}^n)) \hookrightarrow H_T^*(\tilde{G}_k(\mathbb{R}^n))$$

are both injective. Moreover,

$$\pi^*(H^*(G_k(\mathbb{R}^n))) = H^*(\tilde{G}_k(\mathbb{R}^n))^{\mathbb{Z}/2} \quad \text{and} \quad \pi^*(H_T^*(G_k(\mathbb{R}^n))) = H_T^*(\tilde{G}_k(\mathbb{R}^n))^{\mathbb{Z}/2}$$

identifies cohomologies of real Grassmannian as the  $\mathbb{Z}/2$ -invariant subrings of cohomologies of oriented Grassmannian, or equivalently as the  $+1$ -eigenspace of  $\rho^*$  on cohomologies of oriented Grassmannian.

For odd dimensional oriented Grassmannian  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ , the deck transformation  $\rho : \tilde{G}_{2k+1}(\mathbb{R}^{2n+2}) \rightarrow \tilde{G}_{2k+1}(\mathbb{R}^{2n+2}) : v_1 \wedge \cdots \wedge v_{2k+1} \mapsto -(v_1 \wedge \cdots \wedge v_{2k+1}) = (-v_1) \wedge \cdots \wedge (-v_{2k+1})$  is induced from the antipodal map  $A : \mathbb{R}^{2n+2} \rightarrow \mathbb{R}^{2n+2} : v \mapsto -v$  which is homotopic to the identity map on  $\mathbb{R}^{2n+2}$  via

$$\begin{pmatrix} \cos \theta & -\sin \theta & & & \\ \sin \theta & \cos \theta & & & \\ & & \ddots & & \\ & & & \cos \theta & -\sin \theta \\ & & & \sin \theta & \cos \theta \end{pmatrix}$$

which is actually  $T^n$ -equivariant and further induces  $T^n$ -equivariant homotopy between  $\rho$  and  $id$  on  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ .

**Theorem 6.1.1** (Relations between odd dimensional oriented and real Grassmannians). *Since  $\rho^* = id$  on ordinary and equivariant cohomologies of odd dimensional oriented Grassmannian, we have*

$$\begin{aligned} H^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})) &\cong H^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}))^{\mathbb{Z}/2} = H^*(G_{2k+1}(\mathbb{R}^{2n+2})) \\ H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})) &\cong H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}))^{\mathbb{Z}/2} = H_T^*(G_{2k+1}(\mathbb{R}^{2n+2})) \end{aligned}$$

**Corollary 6.1.1.** *The Poincaré series of  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$  are*

$$P_{\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})}(t) = P_{G_{2k+1}(\mathbb{R}^{2n+2})}(t) = (1 + t^{2n+1})P_{G_k(\mathbb{C}^n)}(t^2)$$

The relations between cohomologies of oriented and real Grassmannians in even dimensions is more delicate. In next two subsections, we will try to understand the  $\rho^*$ -action on finer structures of the cohomology ring of oriented Grassmannian.

**6.2. GKM description of oriented Grassmannian.** The GKM description of oriented Grassmannian is very similar to that of real Grassmannian in previous section. Hence most of the details will be omitted but referred to those of real Grassmannian.

**6.2.1. Equivariant Euler classes of canonical bundle and complementary bundle.** The preferred orientation on every oriented  $k$ -dimensional subspace in  $\mathbb{R}^n$  brings new invariants.

For example, the  $T^n$ -fixed points of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  as  $2k$ -dimensional  $T^n$ -subrepresentation of  $\mathbb{R}^{2n} = \oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2$  are of the form  $\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2$  where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . Though an orientation on  $\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2$  can not specify the signs of the individual weights  $\alpha_i, i \in S$ , it specifies the sign of the product of weights as either  $\prod_{i \in S} \alpha_i$  or  $-\prod_{i \in S} \alpha_i$ , which is exactly the equivariant Euler class of an oriented  $T$ -representation over a point. We will denote  $V_{S+}, V_{S-}$  for the  $T$ -subrepresentation  $\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2$  with equivariant Euler classes  $e^T$  as  $\prod_{i \in S} \alpha_i, -\prod_{i \in S} \alpha_i$  respectively. Similarly, for  $\tilde{G}_{2k}((\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0)$  we can introduce the same notations with the fixed points  $V_{S\pm} = (\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2, \pm \prod_{i \in S} \alpha_i)$ .

For  $\tilde{G}_{2k+1}((\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0)$ , the fixed points are of the form  $(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0$  which has equivariant Euler class 0 because of the 0-weight space  $\mathbb{R}_0$ . But an orientation on  $(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0$  gives the complementary  $T^n$ -subrepresentation  $\oplus_{j \notin S} \mathbb{R}_{[\alpha_j]}^2$  an orientation hence an equivariant Euler class  $\bar{e}^T$  either  $\prod_{j \notin S} \alpha_j$  or  $-\prod_{j \notin S} \alpha_j$ . We will denote these fixed points as  $V_{S\pm} = ((\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0, \pm \prod_{j \notin S} \alpha_j)$ .

For  $\tilde{G}_{2k+1}((\oplus_{i=1}^n \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0^2)$ , a fixed component is of the form  $\tilde{C}_S = \{(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0 \mid L_0 \in \tilde{G}_1(\mathbb{R}_0^2)\} \cong S^1$ . Since both  $(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0$  and its complement  $(\oplus_{j \notin S} \mathbb{R}_{[\alpha_j]}^2) \oplus L_0^\perp$  have a 0-weight part, the equivariant Euler classes of both two  $T^n$ -subrepresentations are 0.

**6.2.2. 1-skeleta.** We can describe the 1-skeleta of oriented Grassmannians:

**Proposition 6.2.1** (1-skeleton of oriented Grassmannian). *The fixed points, isotropy weights and 1-skeletons of oriented Grassmannians can be given as*

- (1) For  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ , there are  $2\binom{n}{k}$  fixed points of the form  $V_{S\pm} = (\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2, \pm \prod_{i \in S} \alpha_i)$ , where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . The isotropy weights at both  $V_{S\pm}$  are  $\{[\alpha_j \pm \alpha_i] \mid i \in S, j \notin S\}$ , among which  $[\alpha_j - \alpha_i]$  joins  $V_{S\pm}$  via a 2-sphere to  $V_{((S \setminus \{i\}) \cup \{j\})_{\pm}}$  and  $[\alpha_j + \alpha_i]$  joins  $V_{S\pm}$  via a 2-sphere to  $V_{((S \setminus \{i\}) \cup \{j\})_{\mp}}$ .
- (2)  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ , there are  $2\binom{n}{k}$  fixed points of the form  $V_{S\pm} = (\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2, \pm \prod_{i \in S} \alpha_i)$ , where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . The isotropy weights at both  $V_{S\pm}$  are  $\{[\alpha_j \pm \alpha_i] \mid i \in S, j \notin S\} \cup \{[\alpha_i] \mid i \in S\}$ , among which  $[\alpha_j - \alpha_i]$  joins  $V_{S\pm}$  via a 2-sphere to  $V_{((S \setminus \{i\}) \cup \{j\})_{\pm}}$  and  $[\alpha_j + \alpha_i]$  joins  $V_{S\pm}$  via a 2-sphere to  $V_{((S \setminus \{i\}) \cup \{j\})_{\mp}}$ , and  $[\alpha_i]$  joins  $V_{S\pm}$  via a 2-sphere to  $V_{S_{\mp}}$ .
- (3) For  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ , there are  $2\binom{n}{k}$  fixed points of the form  $V_{S\pm} = ((\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus \mathbb{R}_0, \pm \prod_{j \notin S} \alpha_j)$ , where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . The isotropy weights at both  $V_{S\pm}$  are  $\{[\alpha_j \pm \alpha_i] \mid i \in S, j \notin S\} \cup \{[\alpha_j] \mid j \notin S\}$ , among which  $[\alpha_j - \alpha_i]$  joins  $V_{S\pm}$  via a 2-sphere to  $V_{((S \setminus \{i\}) \cup \{j\})_{\pm}}$  and  $[\alpha_j + \alpha_i]$  joins  $V_{S\pm}$  via a 2-sphere to  $V_{((S \setminus \{i\}) \cup \{j\})_{\mp}}$ , and  $[\alpha_j]$  joins  $V_{S\pm}$  via a 2-sphere to  $V_{S_{\mp}}$ .
- (4) For  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ , there are  $\binom{n}{k}$  fixed circles of the form  $\tilde{C}_S = \{(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0 \mid L_0 \in \tilde{G}_1(\mathbb{R}_0^2)\} \cong S^1$ , where  $S$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$ . The isotropy weights at  $\tilde{C}_S$  are  $\{[\alpha_j \pm \alpha_i] \mid i \in S, j \notin S\} \cup \{[\alpha_i] \mid i \in S\} \cup \{[\alpha_j] \mid j \notin S\}$ , among which both  $[\alpha_j + \alpha_i]$  and  $[\alpha_j - \alpha_i]$  join  $\tilde{C}_S$  via a  $S^2 \times S^1$  to  $\tilde{C}_{(S \setminus \{i\}) \cup \{j\}}$ , and  $[\alpha_i], [\alpha_j]$  join  $\tilde{C}_S$  via a  $S^3$  to no other fixed circles.

*Proof.* Similar to the case of real Grassmannians. □

**6.2.3. GKM graphs of oriented Grassmannians.** Using the 1-skeleta of oriented Grassmannians, we can construct their GKM graphs:

- (1) For oriented Grassmannian  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ , the GKM graph is based on the doubling of Johnson graph: a vertex  $S_{\pm}$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$  with sign  $\pm$ ; two different vertices  $S_+, S'_+$  or  $S_+, S'_-$  with  $S \cup \{j\} = S' \cup \{i\}$  are joined by an edge with weights  $[\alpha_j - \alpha_i]$  and  $[\alpha_j + \alpha_i]$  respectively.
  - (2) For oriented Grassmannian  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ , the GKM graph is based on the doubling of Johnson graph: a vertex  $S_{\pm}$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$  with sign  $\pm$ ; two different vertices  $S_+, S'_+$  or  $S_+, S'_-$  with  $S \cup \{j\} = S' \cup \{i\}$  are joined by an edge with weights  $[\alpha_j - \alpha_i]$  and  $[\alpha_j + \alpha_i]$  respectively; the twin vertices  $S_+, S_-$  are joined by  $k$  edges with weights  $\alpha_i, i \in S$  respectively.
  - (3) For oriented Grassmannian  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ , the GKM graph is based on the doubling of Johnson graph: a vertex  $S_{\pm}$  is a  $k$ -element subset of  $\{1, 2, \dots, n\}$  with sign  $\pm$ ; two different vertices  $S_+, S'_+$  or  $S_+, S'_-$  with  $S \cup \{j\} = S' \cup \{i\}$  are joined by an edge with weights  $[\alpha_j - \alpha_i]$  and  $[\alpha_j + \alpha_i]$  respectively; the twin vertices  $S_+, S_-$  are joined by  $n - k$  edges with weights  $\alpha_j, j \notin S$  respectively.
  - (4) For odd dimensional Grassmannian  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ , the GKM graph consists of
    - **vertices:** For each  $k$ -element subset  $S \subseteq \{1, 2, \dots, n\}$ , there is a  $\circ$ .
    - ◻ **vertices, weights:** For each pair of different vertices  $S, S'$  with  $S \cup \{j\} = S' \cup \{i\}$ , there are two  $\square$ 's weighted  $[\alpha_j - \alpha_i]$  and  $[\alpha_j + \alpha_i]$  respectively. Near each  $S$ , there are  $n$  extra  $\square$ 's weighted  $[\alpha_1], \dots, [\alpha_n]$  respectively.
- Edges:** For each  $\circ$  with symbol  $S$ , we join it to all the nearby  $\square$  vertices of its isotropy weights.

#### 6.2.4. Formality, cohomology and canonical basis of oriented Grassmannians.

**Proposition 6.2.2** (Equivariant formality of torus actions on oriented Grassmannians). *The  $T^n$ -actions on all the four types of oriented Grassmannians  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ ,  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ ,  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ ,  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$  are equivariantly formal. All the four oriented Grassmannians have the same total Betti number  $2^{\binom{n}{k}}$ .*

*Proof.* The even dimensional oriented Grassmannians  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ ,  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ ,  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$  can be viewed as homogeneous spaces of the form  $G/H$  with  $H$  a compact connected Lie subgroup and of the same rank as the connected compact Lie group  $G$ . According to [GHZ06], these homogeneous spaces are equivariantly formal. Their total Betti numbers are the same as their Euler characteristic numbers  $|W_G/W_H|$  where  $W_G, W_H$  are the Weyl groups of  $G$  and  $H$ . Alternatively, we can compute the total Betti number as the number of fixed points given in Theorem 6.2.1, namely  $2^{\binom{n}{k}}$ . The odd dimensional oriented Grassmannian  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$  has the same equivariant cohomology as the real Grassmannian  $G_{2k+1}(\mathbb{R}^{2n+2})$ , hence also equivariantly formal with total Betti number  $2^{\binom{n}{k}}$ .  $\square$

After verifying GKM conditions and equivariant formality, we can give the GKM description of the torus actions on  $\tilde{G}_k(\mathbb{R}^n)$  by applying the classical even dimensional GKM Theorem as in Guillemin, Holm & Zara [GHZ06] and the odd dimensional GKM Theorem 3.3.1.

**Theorem 6.2.1** (GKM description of oriented Grassmannians). *The following congruence relations are given for any two  $k$ -element subsets  $S, S' \subset \{1, \dots, n\}$  differed by one element with  $S \cup \{j\} = S' \cup \{i\}$ .*

- (1) For the oriented Grassmannian  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ , an element of equivariant cohomology is a set of polynomials  $f_{S_{\pm}} \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$  to each vertex  $S_{\pm}$  such that
  - (a)  $f_{S_+} \equiv f_{S'_+} \pmod{\alpha_j - \alpha_i}$
  - (b)  $f_{S_+} \equiv f_{S'_-} \pmod{\alpha_j + \alpha_i}$
- (2) For the oriented Grassmannian  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ , an element of equivariant cohomology is a set of polynomials  $f_{S_{\pm}} \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$  to each vertex  $S_{\pm}$  such that
  - (a)  $f_{S_+} \equiv f_{S'_+} \pmod{\alpha_j - \alpha_i}$
  - (b)  $f_{S_+} \equiv f_{S'_-} \pmod{\alpha_j + \alpha_i}$
  - (c)  $f_{S_+} \equiv f_{S_-} \pmod{\prod_{i' \in S} \alpha_{i'}}$
- (3) For the oriented Grassmannian  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ , an element of equivariant cohomology is a set of polynomials  $f_{S_{\pm}} \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$  to each vertex  $S_{\pm}$  such that

- (a)  $f_{S_+} \equiv f_{S'_+}, \quad f_{S_-} \equiv f_{S'_-} \pmod{\alpha_j - \alpha_i}$
- (b)  $f_{S_+} \equiv f_{S'_-} \pmod{\alpha_j + \alpha_i}$
- (c)  $f_{S_+} \equiv f_{S_-} \pmod{\prod_{j' \neq S} \alpha_{j'}}$
- (4) For the odd dimensional oriented Grassmannian  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$  with  $T^n$ -action, an element of equivariant cohomology is a set of polynomial pairs  $(f_S, g_S \theta)$  to each  $\circ$ -vertex  $S$  where  $\theta$  is the unit volume form of  $S^1$  such that
  - (a)  $g_S \equiv 0 \pmod{\prod_{i=1}^n \alpha_i}$
  - (b)  $f_S \equiv f_{S'}, \quad g_S \equiv g_{S'} \pmod{\alpha_j^2 - \alpha_i^2}$

For the odd dimensional oriented Grassmannian, the induced deck transformation  $\rho^* : H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})) \rightarrow H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}))$  is the identity map hence acts trivially on the GKM description. Solving the same set of congruence equations as in Theorem 5.1.2 of  $H_T^*(G_{2k+1}(\mathbb{R}^{2n+2}))$ , we will also get an element  $\tilde{r}^T \in H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}))$  in GKM description localized at a fixed circle  $\tilde{C}_S = \{(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0 \mid L_0 \in \tilde{G}_1(\mathbb{R}_0^2)\} \cong S^1$  to be  $\tilde{r}_S^T = \prod_{i=1}^n \alpha_i \theta_{S^1}$  similar to the  $r^T \in H_T^*(G_{2k+1}(\mathbb{R}^{2n+2}))$  localized at  $C_S = \{(\oplus_{i \in S} \mathbb{R}_{[\alpha_i]}^2) \oplus L_0 \mid L_0 \in G_1(\mathbb{R}_0^2)\} \cong \mathbb{R}P^1$  to be  $r_S^T = \prod_{i=1}^n \alpha_i \theta_{\mathbb{R}P^1}$ , where  $\theta_{S^1}$  and  $\theta_{\mathbb{R}P^1}$  are the unit volume forms of  $S^1$  and  $\mathbb{R}P^1$  respectively.

**Proposition 6.2.3** (Canonical basis of  $H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}))$ ). *Let  $\sigma_S \in \mathcal{S}$  be the canonical basis of  $H_T^*(G_{2k}(\mathbb{R}^{2n}))$  from Theorem 5.1.3 and  $\tilde{r}_S^T = \prod_{i=1}^n \alpha_i \theta_{S^1}$  be the odd-degree generator. Then  $\sigma_S, \tilde{r}_S^T \cdot \sigma_S$  give additive  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -basis of  $H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}))$ .*

However, there is a subtlety for the pullback  $\pi^* : H_T^*(G_{2k+1}(\mathbb{R}^{2n+2})) \rightarrow H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}))$  though this is an isomorphism. The 2-fold covering  $\pi : \tilde{G}_{2k+1}(\mathbb{R}^{2n+2}) \rightarrow G_{2k+1}(\mathbb{R}^{2n+2})$  restricts to a 2-fold covering of fixed circles  $\pi : (\tilde{C}_S \cong S^1) \rightarrow (C_S \cong \mathbb{R}P^1)$  which will give the localized pullback  $\pi^*(\theta_{\mathbb{R}P^1}) = 2\theta_{S^1}$ . Hence we get  $\pi^*(r^T) = 2\tilde{r}^T$ .

**Proposition 6.2.4** (The explicit pullback of cohomology between odd dimensional Grassmannians). *In the canonical basis, the pullback of cohomology of odd dimensional Grassmannian is*

$$\begin{aligned} \pi^* : H_T^*(G_{2k+1}(\mathbb{R}^{2n+2})) &\longrightarrow H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})) \\ \sigma_S &\longmapsto \sigma_S \\ r^T \cdot \sigma_S &\longmapsto 2\tilde{r}^T \cdot \sigma_S \end{aligned}$$

For the even dimensional oriented Grassmannians, the deck transformation  $\rho : \tilde{G}_k(\mathbb{R}^n) \rightarrow \tilde{G}_k(\mathbb{R}^n)$  switches any fixed point  $p_{S_+}$  with its twin fixed point  $p_{S_-}$  by reversing orientations. Then the induced deck transformation  $\rho^* : H_T^*(\tilde{G}_k(\mathbb{R}^n)) \rightarrow H_T^*(\tilde{G}_k(\mathbb{R}^n))$  in GKM description will switch any polynomial  $f_{S_+}$  with  $f_{S_-}$ . Notice the symmetry in the GKM descriptions, we see that the switch of polynomials preserves the congruence relations.

Since  $(\rho^*)^2 = id$ , both two cohomologies  $H^*(\tilde{G}_k(\mathbb{R}^n)), H_T^*(\tilde{G}_k(\mathbb{R}^n))$  decompose into  $\pm 1$ -eigenspaces of  $\rho^*$ .

**Proposition 6.2.5.** *For the even dimensional oriented Grassmannians  $\tilde{G}_{2k}(\mathbb{R}^{2n}), \tilde{G}_{2k}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ , the elements of  $+1$ -eigenspace of  $\rho^*$  can be identified as those sets of polynomials  $\{f_{S_{\pm}}, S \in \mathcal{S}\}$  where  $\mathcal{S}$  is the collection of  $k$ -element subsets of  $\{1, \dots, n\}$  such that*

$$f_{S_+} = f_{S_-}$$

*and the elements of  $-1$ -eigenspace of  $\rho^*$  are those with*

$$f_{S_+} = -f_{S_-}$$

**Remark 6.2.1.** We have shown in Proposition 6.1.1 that the  $+1$ -eigenspaces of  $\rho^*$  on the equivariant cohomology of oriented Grassmannians are exactly the equivariant cohomology of real Grassmannians. We can reprove this by plugging  $f_{S_+} = f_{S_-}$  into the GKM descriptions of even dimensional oriented Grassmannians and condensing the congruence relations to be the same as those of the even dimensional real Grassmannians.



Recall that we defined equivariant Euler classes at each fixed point  $S$  for  $\tilde{G}_{2k}(\mathbb{R}^{2n}), \tilde{G}_{2k}(\mathbb{R}^{2n+1})$  to be  $e_{S_{\pm}}^T = \pm \prod_{i \in S} \alpha_i$  and for  $\tilde{G}_{2k}(\mathbb{R}^{2n}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$  to be  $\bar{e}_{S_{\pm}}^T = \pm \prod_{j \notin S} \alpha_j$ . It is easy to check that  $\{e_{S_{\pm}}^T, S \in \mathcal{S}\}$  and  $\{\bar{e}_{S_{\pm}}^T, S \in \mathcal{S}\}$  are elements of the GKM description of the corresponding equivariant cohomology. Since  $\rho$  changes the signs of orientations,  $\rho^*$  changes the signs of the equivariant Euler classes, i.e.  $\{e_{S_{\pm}}^T, S \in \mathcal{S}\}$  and  $\{\bar{e}_{S_{\pm}}^T, S \in \mathcal{S}\}$  are in the  $-1$ -eigenspaces of  $\rho^*$ . Geometrically, the equivariant Euler classes  $e^T, \bar{e}^T$  in GKM description are exactly the equivariant Euler classes of the canonical oriented bundles and complementary oriented bundles over the oriented Grassmannians.

**Proposition 6.2.6** (Equivariant Euler class and top equivariant Pontryagin class). *Similar to the relations between ordinary Euler class and top ordinary Pontryagin class,*

- (1) For  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  and  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ , we have  $(e^T)^2 = p_k^T$
- (2) For  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  and  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ , we have  $(\bar{e}^T)^2 = \bar{p}_{n-k}^T$
- (3) For  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ , we have  $e^T \bar{e}^T = \prod_{i=1}^n \alpha_i$

*Proof.* Let's prove this for  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  which covers the remaining cases of  $\tilde{G}_{2k}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ . In Proposition 5.2.1, we have given the localized top equivariant Pontryagin classes of real Grassmannian as

$$p_k^T|_S = \prod_{i \in S} \alpha_i^2 \quad \bar{p}_{n-k}^T|_S = \prod_{j \notin S} \alpha_j^2$$

Via the pullback  $\pi^* : H_T^*(G_{2k}(\mathbb{R}^{2n})) \rightarrow H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n}))$ , the equivariant Pontryagin classes of  $G_{2k}(\mathbb{R}^{2n})$  are identified as those of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ , and are in the  $+1$ -eigenspaces of  $\rho^*$ . Therefore

$$p_k^T|_{S_{\pm}} = \prod_{i \in S} \alpha_i^2 \quad \bar{p}_{n-k}^T|_{S_{\pm}} = \prod_{j \notin S} \alpha_j^2$$

Comparing them with

$$e_{S_{\pm}}^T = \pm \prod_{i \in S} \alpha_i \quad \bar{e}_{S_{\pm}}^T = \pm \prod_{j \notin S} \alpha_j$$

we get the stated relations.  $\square$

The induced deck transformation  $\rho^*$  is a ring homomorphism, therefore the multiplication of an element in the  $-1$ -eigenspace with one in the  $+1$ -eigenspace results in the  $-1$ -eigenspace.

**Proposition 6.2.7.** *Multiplication with the equivariant Euler classes  $e^T, \bar{e}^T$  maps  $+1$ -eigenspaces of  $\rho^*$  to  $-1$ -eigenspaces.*

- (1) For  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ , the multiplication with  $e^T$  is an isomorphism between  $+1$ -eigenspace of  $\rho^*$  to its  $-1$ -eigenspace.
- (2) For  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ , the multiplication with  $\bar{e}^T$  is an isomorphism between  $+1$ -eigenspace of  $\rho^*$  to its  $-1$ -eigenspace.

*Proof.* For  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ , denote  $V_{+1}$  and  $V_{-1}$  be the  $+1$  and  $-1$ -eigenspaces of  $\rho^*$  on  $H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1}))$ . The fact that  $e^T$  is in the  $-1$ -eigenspace gives the multiplication  $\times e^T : V_{+1} \rightarrow V_{-1}$ . On the other hand, every element  $\{f_{S_{\pm}}, S \in \mathcal{S}\}$  of  $V_{-1}$  has the form  $f_{S_{+}} = -f_{S_{-}}$  by Prop 6.2.5. Plug this into the congruence relation between  $S_{+}$  and  $S_{-}$  in Theorem 6.2.1, we get

$$f_{S_{+}} \equiv f_{S_{-}} = -f_{S_{+}} \pmod{\prod_{i \in S} \alpha_i}$$

or equivalently, both  $f_{S_{+}}$  and  $f_{S_{-}}$  are multiples of  $e_{S_{\pm}}^T = \pm \prod_{i \in S} \alpha_i$ . Therefore, the localized quotients  $f_{S_{+}}/e_{S_{+}}^T, f_{S_{-}}/e_{S_{-}}^T \in \mathbb{Q}[\alpha_1, \dots, \alpha_n]$  are polynomials, and this defines a unique element  $f/e^T \in V_{+1}$ .

The case of  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$  is similar.  $\square$

*Remark 6.2.2.* For  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ , neither the multiplication by  $e^T$  nor by  $\bar{e}^T$  are isomorphisms between the  $+1$  and  $-1$ -eigenspaces of  $\rho^*$ . We will try to understand the equivariant cohomology of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  in next subsection.

The above isomorphism between eigenspaces of  $\rho^*$ , together with the canonical basis  $\sigma_S$  of  $H_T^*(G_{2k}(\mathbb{R}^{2n+1}))$  and  $H_T^*(G_{2k+1}(\mathbb{R}^{2n+1}))$ , gives

**Proposition 6.2.8** (Canonical basis of  $H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1}))$ ,  $H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+1}))$ ). *Let  $\sigma_{S \in \mathcal{S}}$  be the canonical basis of  $H_T^*(G_{2k}(\mathbb{R}^{2n+1}))$  and  $H_T^*(G_{2k+1}(\mathbb{R}^{2n+1}))$  from Theorem 5.1.3. Then  $\sigma_S, e^T \cdot \sigma_S$  and  $\sigma_S, \bar{e}^T \cdot \sigma_S$  give additive  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -basis of  $H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1}))$  and  $H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+1}))$  respectively.*

**Corollary 6.2.1.** *The Poincaré series of  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$  and  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$  are*

$$\begin{aligned} P_{\tilde{G}_{2k}(\mathbb{R}^{2n+1})}(t) &= (1 + t^{2k})P_{G_{2k}(\mathbb{R}^{2n+1})}(t) = (1 + t^{2k})P_{G_k(\mathbb{C}^n)}(t^2) \\ P_{\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})}(t) &= (1 + t^{2n-2k})P_{G_{2k+1}(\mathbb{R}^{2n+1})}(t) = (1 + t^{2n-2k})P_{G_k(\mathbb{C}^n)}(t^2) \end{aligned}$$

**Theorem 6.2.2** (Relations between oriented and real Grassmannians). *The equivariant cohomologies of  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$  and  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$  are  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -algebra extensions by  $e^T, \bar{e}^T$  of the equivariant cohomologies of  $G_{2k}(\mathbb{R}^{2n+1})$  and  $G_{2k+1}(\mathbb{R}^{2n+1})$ , i.e.*

$$\begin{aligned} H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1})) &\cong \frac{H_T^*(G_{2k}(\mathbb{R}^{2n+1}))[e^T]}{(e^T)^2 = p_k^T} \\ H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})) &\cong \frac{H_T^*(G_{2k+1}(\mathbb{R}^{2n+1}))[\bar{e}^T]}{(\bar{e}^T)^2 = \bar{p}_{n-k}^T} \end{aligned}$$

*Proof.* First of all, we can define  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -algebra morphisms

$$\begin{aligned} \frac{H_T^*(G_{2k}(\mathbb{R}^{2n+1}))[e^T]}{(e^T)^2 = p_k^T} &\longrightarrow H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1})) \\ \frac{H_T^*(G_{2k+1}(\mathbb{R}^{2n+1}))[\bar{e}^T]}{(\bar{e}^T)^2 = \bar{p}_{n-k}^T} &\longrightarrow H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})) \end{aligned}$$

by sending equivariant Pontryagin classes of real Grassmannian to the corresponding equivariant Pontryagin classes of oriented Grassmannian, and the abstract symbols  $e^T$  or  $\bar{e}^T$  to the actual equivariant Euler classes of canonical bundle or complementary bundle.

Secondly, let  $\sigma_{S \in \mathcal{S}}$  be the canonical basis of  $H_T^*(G_{2k}(\mathbb{R}^{2n+1}))$  and  $H_T^*(G_{2k+1}(\mathbb{R}^{2n+1}))$ ,  $\sigma_S, e^T \cdot \sigma_S$  and  $\sigma_S, \bar{e}^T \cdot \sigma_S$  give the additive  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -basis of  $H_T^*(G_{2k}(\mathbb{R}^{2n+1}))[e^T]/(e^T)^2 = p_k^T$  and  $H_T^*(G_{2k+1}(\mathbb{R}^{2n+1}))[\bar{e}^T]/(\bar{e}^T)^2 = \bar{p}_{n-k}^T$  respectively. However, according to the Prop 6.2.8,  $H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1}))$  and  $H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+1}))$  also have the same additive  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -basis.  $\square$

**6.3. Borel description of oriented Grassmannian.** In this subsection, we will confirm the ring generators of equivariant cohomology of oriented Grassmannian to be characteristic classes, then determine the complete relations among them, and also give additive basis.

**6.3.1. Borel description of  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}), \tilde{G}_{2k}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ .** From Theorem 6.1.1 and Theorem 6.2.2, we have seen that equivariant cohomologies of  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}), \tilde{G}_{2k}(\mathbb{R}^{2n+1})$  and  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$  are ring extensions of the equivariant cohomologies of their real counterparts. Hence the equivariant Borel descriptions and equivariant characteristic basis of those oriented Grassmannians can be extended from the related real Grassmannians.

**Theorem 6.3.1** (Equivariant Borel description of  $\tilde{G}_{2k}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ ). *The equivariant cohomologies of  $\tilde{G}_{2k}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$  are generated by equivariant Pontryagin classes and*

equivariant Euler classes:

$$\begin{aligned} H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1})) &\cong \frac{\mathbb{Q}[\alpha_1, \alpha_2, \dots, \alpha_n][p_1^T, p_2^T, \dots, p_k^T; \bar{p}_1^T, \bar{p}_2^T, \dots, \bar{p}_{n-k}^T; e^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2), (e^T)^2 = p_k^T} \\ H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})) &\cong \frac{\mathbb{Q}[\alpha_1, \alpha_2, \dots, \alpha_n][p_1^T, p_2^T, \dots, p_k^T; \bar{p}_1^T, \bar{p}_2^T, \dots, \bar{p}_{n-k}^T; \bar{e}^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2), (\bar{e}^T)^2 = \bar{p}_{n-k}^T} \\ H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})) &\cong \frac{\mathbb{Q}[\alpha_1, \alpha_2, \dots, \alpha_n][p_1^T, p_2^T, \dots, p_k^T; \bar{p}_1^T, \bar{p}_2^T, \dots, \bar{p}_{n-k}^T; \tilde{r}^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2), (\tilde{r}^T)^2 = 0} \end{aligned}$$

**Theorem 6.3.2** (Equivariant characteristic basis of  $\tilde{G}_{2k}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ ). *The sets of monomials  $\{(p_1^T)^{r_1} (p_2^T)^{r_2} \dots (p_k^T)^{r_k}, e^T \cdot (p_1^T)^{r_1} (p_2^T)^{r_2} \dots (p_k^T)^{r_k}\}$ ,  $\{(p_1^T)^{r_1} (p_2^T)^{r_2} \dots (p_k^T)^{r_k}, \bar{e}^T \cdot (p_1^T)^{r_1} (p_2^T)^{r_2} \dots (p_k^T)^{r_k}\}$  and  $\{(p_1^T)^{r_1} (p_2^T)^{r_2} \dots (p_k^T)^{r_k}, \tilde{r}^T \cdot (p_1^T)^{r_1} (p_2^T)^{r_2} \dots (p_k^T)^{r_k}\}$  satisfying the condition  $\sum_{i=1}^k r_i \leq n - k$  form additive  $H_T^*(pt)$ -basis for  $H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1})), H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})), H_T^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}))$  respectively.*

The above two theorems of equivariant ring generators and equivariant additive basis both have their ordinary versions by replacing  $\alpha_i$  with 0.

**Corollary 6.3.1** (Ordinary Borel description of  $\tilde{G}_{2k}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ ). *The ordinary cohomologies of  $\tilde{G}_{2k}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$  are generated by Pontryagin classes and Euler classes:*

$$\begin{aligned} H^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1})) &\cong \frac{\mathbb{Q}[p_1, p_2, \dots, p_k; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{n-k}; e]}{p\bar{p} = 1, e^2 = p_k} \\ H^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})) &\cong \frac{\mathbb{Q}[p_1, p_2, \dots, p_k; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{n-k}; \bar{e}]}{p\bar{p} = 1, \bar{e}^2 = \bar{p}_{n-k}} \\ H^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})) &\cong \frac{\mathbb{Q}[p_1, p_2, \dots, p_k; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{n-k}; \tilde{r}]}{p\bar{p} = 1, \tilde{r}^2 = 0} \end{aligned}$$

**Corollary 6.3.2** (Ordinary characteristic basis of  $\tilde{G}_{2k}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ ). *The sets of monomials  $\{p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}, e \cdot p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}\}$ ,  $\{p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}, \bar{e} \cdot p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}\}$  and  $\{p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}, \tilde{r} \cdot p_1^{r_1} p_2^{r_2} \dots p_k^{r_k}\}$  satisfying the condition  $\sum_{i=1}^k r_i \leq n - k$  form additive basis for  $H^*(\tilde{G}_{2k}(\mathbb{R}^{2n+1})), H^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})), H^*(\tilde{G}_{2k+1}(\mathbb{R}^{2n+2}))$  respectively.*

**6.3.2. Borel description of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ .** Now let's turn to the remaining type of oriented Grassmannian  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ . As we remarked in previous subsection, neither the multiplication by  $e^T$  nor by  $\bar{e}^T$  are isomorphisms between eigenspaces of  $\rho^*$ . However, we will show the multiplications by  $e^T$  and  $\bar{e}^T$ , restricted on certain carefully chosen subspaces, do give isomorphism between  $+1$  and  $-1$ -eigenspaces of  $\rho^*$ .

Notice the equivariant diffeomorphism  $G_{2k}(\mathbb{R}^{2n}) \cong G_{2n-2k}(\mathbb{R}^{2n})$  by mapping an oriented  $2k$ -dimensional subspace to its perpendicular oriented  $(2n - 2k)$ -dimensional subspace. Then the complementary characteristic monomials  $(\bar{p}_1^T)^{r_1} (\bar{p}_2^T)^{r_2} \dots (\bar{p}_{n-k}^T)^{r_{n-k}}$  and  $\bar{p}_1^{r_1} \bar{p}_2^{r_2} \dots \bar{p}_{n-k}^{r_{n-k}}$ , satisfying the condition  $\sum_{i=1}^{n-k} r_i \leq k$ , give additive basis for the equivariant and respectively ordinary cohomology of  $G_{2k}(\mathbb{R}^{2n})$ . Also recall from Prop 6.2.6 on the relations among top Pontryagin classes and Euler classes of the oriented Grassmannian  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  that  $(e^T)^2 = p_k^T, (\bar{e}^T)^2 = \bar{p}_{n-k}^T, e^T \bar{e}^T = \prod_{i=1}^n \alpha_i$  and  $e^2 = p_k, \bar{e}^2 = \bar{p}_{n-k}, e\bar{e} = 0$ .

**Proposition 6.3.1** (Eigenspaces of  $H^*(\tilde{G}_{2k}(\mathbb{R}^{2n}))$ ). *Let  $\rho$  be the non-trivial deck transformation of the covering  $\pi : \tilde{G}_{2k}(\mathbb{R}^{2n}) \rightarrow G_{2k}(\mathbb{R}^{2n})$  and identify  $H^*(G_{2k}(\mathbb{R}^{2n}))$  as the  $+1$ -eigenspace of  $\rho^*$  on  $H^*(\tilde{G}_{2k}(\mathbb{R}^{2n}))$ .*

- (1) The multiplications by  $e^T$  and  $\bar{e}^T$  are isomorphisms restricted on the following subspaces

$$e \times : \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1) \xrightarrow{\cong} e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1)$$

$$\bar{e} \times : \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k - 1) \xrightarrow{\cong} \bar{e} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k - 1)$$

- (2)  $e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1) \oplus \bar{e} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k - 1)$  is the  $(-1)$ -eigenspace of  $\rho^*$   
(3)  $e \cdot H^*(G_{2k}(\mathbb{R}^{2n})) \cap \bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n})) = 0$  and  $e \cdot H^*(G_{2k}(\mathbb{R}^{2n})) \oplus \bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$  is the  $(-1)$ -eigenspace of  $\rho^*$   
(4) The kernels of  $e \times$  and  $\bar{e} \times$  on  $H^*(G_{2k}(\mathbb{R}^{2n}))$  are  $\bar{p}_{n-k} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k - 1)$  and  $p_k \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1)$  respectively  
(5) The following spaces are identical

$$e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1) = e \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k-1}^{r_{n-k-1}} \mid \sum_{i=1}^{n-k-1} r_i \leq k) = e \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$$

$$\bar{e} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k - 1) = \bar{e} \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} \mid \sum_{i=1}^{k-1} r_i \leq n - k) = \bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$$

*Proof.* Note that the total Betti numbers of  $H^*(G_{2k}(\mathbb{R}^{2n}))$  and  $H^*(\tilde{G}_{2k}(\mathbb{R}^{2n}))$  are  $\binom{n}{k}$  and  $2\binom{n}{k}$  respectively, hence the dimension of the  $-1$ -eigenspace of  $\rho^*$  is  $\binom{n}{k}$ .

- (1) The composition of the surjective linear maps

$$e \times : \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1) \longrightarrow e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1)$$

$$e \times : e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1) \longrightarrow e^2 \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1)$$

is

$$p_k \times : \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1) \longrightarrow p_k \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1)$$

where we have used the relation  $e^2 = p_k$ . The composition maps a sub-basis of  $H^*(G_{2k}(\mathbb{R}^{2n}))$  onto another sub-basis without common vectors, hence is a bijection. Therefore, each individual surjection is a bijection. Similarly, we get the bijection for the restricted  $\bar{e} \times$ .

- (2) We have seen from the above that

$$e \times : e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1) \longrightarrow p_k \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1)$$

is a bijection. However,  $e \times$  takes  $\bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$  to zero, because  $e\bar{e} = 0$ . Hence

$$e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k - 1) \cap \bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n})) = 0$$

Similarly,

$$\bar{e} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k-1) \cap e \cdot H^*(G_{2k}(\mathbb{R}^{2n})) = 0$$

Combine these two, we get

$$e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n-k-1) \cap \bar{e} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k-1) = 0$$

However, as a subspace in  $-1$ -eigenspace of  $\rho^*$ , the sum  $e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n-k-1) \oplus \bar{e} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k-1)$  has dimension  $\binom{n-1}{k} + \binom{n-1}{n-k} = \binom{n}{k}$  the same as dimension of the entire  $-1$ -eigenspace of  $\rho^*$ , hence is exactly the  $-1$ -eigenspace of  $\rho^*$ .

- (3) The above series of zero intersections force  $e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n-k-1) = e \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$  and  $\bar{e} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k-1) = \bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$ . Hence we get  $e \cdot H^*(G_{2k}(\mathbb{R}^{2n})) \cap \bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n})) = 0$  and  $e \cdot H^*(G_{2k}(\mathbb{R}^{2n})) \oplus \bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$  is the  $(-1)$ -eigenspace of  $\rho^*$ .
- (4) We have proved  $e \cdot H^*(G_{2k}(\mathbb{R}^{2n})) = e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n-k-1)$  and they are of dimension  $\binom{n-1}{k}$ . Since  $H^*(G_{2k}(\mathbb{R}^{2n}))$  is of dimension  $\binom{n}{k}$ , the kernel of  $e \times$  on  $H^*(G_{2k}(\mathbb{R}^{2n}))$  is then of dimension  $\binom{n}{k} - \binom{n-1}{k} = \binom{n-1}{n-k}$ . Because  $e \cdot \bar{p}_{n-k} = e \cdot \bar{e}^2 = 0$ , the subspace  $\bar{p}_{n-k} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k-1)$  of dimension  $\binom{n-1}{n-k}$  is clearly in the kernel of  $e \times$  on  $H^*(G_{2k}(\mathbb{R}^{2n}))$ , hence is exactly the kernel. Similarly, we obtain the kernel of  $\bar{e} \times$  on  $H^*(G_{2k}(\mathbb{R}^{2n}))$ .
- (5) The  $e \times$ -kernel subspace  $\bar{p}_{n-k} \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k-1)$  of  $H^*(G_{2k}(\mathbb{R}^{2n}))$  has complementary subspace  $\text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k-1}^{r_{n-k-1}} \mid \sum_{i=1}^{n-k-1} r_i \leq k)$ . Hence the restriction

$$e \times : \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k-1}^{r_{n-k-1}} \mid \sum_{i=1}^{n-k-1} r_i \leq k) \longrightarrow e \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$$

is bijection, therefore  $e \cdot \text{Span}(\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k-1}^{r_{n-k-1}} \mid \sum_{i=1}^{n-k-1} r_i \leq k) = e \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$ . The identification  $e \cdot \text{Span}(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n-k-1) = e \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$  is proved in (3). Similarly, we get the identifications for  $\bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$ . □

The detailed discussion of  $e \times$  and  $\bar{e} \times$  between the eigenspaces of  $\rho^*$  gives:

**Corollary 6.3.3** (Ordinary characteristic basis of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ ). *The ordinary cohomology of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  is generated by Pontryagin classes and Euler classes with an additive basis  $\{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n-k\}$  for the  $+1$ -eigenspace of  $\rho^*$  and  $\{e \cdot \bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k-1}^{r_{n-k-1}} \mid \sum_{i=1}^{n-k-1} r_i \leq k\} \cup \{\bar{e} \cdot p_1^{r_1} p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} \mid \sum_{i=1}^{k-1} r_i \leq n-k\}$  for the  $-1$ -eigenspace.*

*Remark 6.3.1.* Using the various identifications of  $e \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$  and  $\bar{e} \cdot H^*(G_{2k}(\mathbb{R}^{2n}))$  in Theorem 6.3.1, we can also give the additive basis of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  in other forms.

**Corollary 6.3.4.** *The Poincaré series of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  are*

$$\begin{aligned} P_{\tilde{G}_{2k}(\mathbb{R}^{2n})}(t) &= P_{G_{2k}(\mathbb{R}^{2n})}(t) + t^{2k} P_{G_{2k}(\mathbb{R}^{2n-2})}(t) + t^{2n-2k} P_{G_{2k-2}(\mathbb{R}^{2n-2})}(t) \\ &= P_{G_k(\mathbb{C}^n)}(t^2) + t^{2k} P_{G_k(\mathbb{C}^{n-1})}(t^2) + t^{2n-2k} P_{G_{k-1}(\mathbb{C}^{n-1})}(t^2) \end{aligned}$$

*Proof.* Notice that the  $-1$ -eigenbasis  $\{e \cdot \bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k-1}^{r_{n-k-1}} \mid \sum_{i=1}^{n-k-1} r_i \leq k\} \cup \{\bar{e} \cdot p_1^{r_1} p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} \mid \sum_{i=1}^{k-1} r_i \leq n-k\}$  has factors  $\{\bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k-1}^{r_{n-k-1}} \mid \sum_{i=1}^{n-k-1} r_i \leq k\}$  and  $\{p_1^{r_1} p_2^{r_2} \cdots p_{k-1}^{r_{k-1}} \mid \sum_{i=1}^{k-1} r_i \leq n-k\}$  which also appear as the additive basis of  $H^*(G_{2k}(\mathbb{R}^{2n-2}))$  and  $H^*(G_{2k-2}(\mathbb{R}^{2n-2}))$  respectively. □

**Theorem 6.3.3** (Ordinary Borel description of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ ). *The ordinary cohomology of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  is a ring extension of the ordinary cohomology of  $G_{2k}(\mathbb{R}^{2n})$ :*

$$H^*(\tilde{G}_{2k}(\mathbb{R}^{2n})) \cong \frac{H^*(G_{2k}(\mathbb{R}^{2n}))[e, \bar{e}]}{e^2 = p_k, \bar{e}^2 = \bar{p}_{n-k}, e\bar{e} = 0} \cong \frac{\mathbb{Q}[p_1, p_2, \dots, p_k; \bar{p}_1, \bar{p}_2, \dots, \bar{p}_{n-k}; e, \bar{e}]}{p\bar{p} = 1, e^2 = p_k, \bar{e}^2 = \bar{p}_{n-k}, e\bar{e} = 0}$$

*Proof.* Consider the ring homomorphism

$$\frac{H^*(G_{2k}(\mathbb{R}^{2n}))[e, \bar{e}]}{e^2 = p_k, \bar{e}^2 = \bar{p}_{n-k}, e\bar{e} = 0} \longrightarrow H^*(\tilde{G}_{2k}(\mathbb{R}^{2n}))$$

which sends Pontryagin classes of  $G_{2k}(\mathbb{R}^{2n})$  to the corresponding Pontryagin classes of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  and sends the abstract symbols  $e, \bar{e}$  to the actual Euler classes of the oriented canonical bundle and complementary bundle over  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ . Since we have proved that  $H^*(\tilde{G}_{2k}(\mathbb{R}^{2n}))$  is generated on Pontryagin classes and Euler classes, the above morphism is surjective. It is easy check that  $H^*(G_{2k}(\mathbb{R}^{2n}))[e, \bar{e}]/\{e^2 = p_k, \bar{e}^2 = \bar{p}_{n-k}, e\bar{e} = 0\}$  also has the same additive basis  $\{p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k\} \cup \{e \cdot \bar{p}_1^{r_1} \bar{p}_2^{r_2} \cdots \bar{p}_{n-k}^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k\} \cup \{\bar{e} \cdot p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k} \mid \sum_{i=1}^k r_i \leq n - k\}$  as  $H^*(\tilde{G}_{2k}(\mathbb{R}^{2n}))$ . Hence, we get a ring isomorphism.  $\square$

*Remark 6.3.2.* The ordinary cohomology of  $\tilde{G}_k(\mathbb{R}^n)$  in  $\mathbb{Z}$  coefficients for  $n \leq 8$  was computed by Junkind [Ju79]. The ordinary cohomology of  $\tilde{G}_k(\mathbb{R}^n)$  in  $\mathbb{R}$  coefficients for  $k = 2$  was computed by Shi&Zhou [SZ14].

Since  $T^n$  acts on  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  equivariantly formal, i.e.  $H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n})) \cong \mathbb{Q}[\alpha_1, \dots, \alpha_n] \otimes_{\mathbb{Q}} H^*(\tilde{G}_{2k}(\mathbb{R}^{2n}))$  as  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -modules, we can lift the ordinary basis, characteristic classes and relations to be equivariant, then obtain the equivariant versions of characteristic basis and Borel description:

**Corollary 6.3.5** (Equivariant Borel description and characteristic basis of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$ ). *The equivariant cohomology of  $\tilde{G}_{2k}(\mathbb{R}^{2n})$  is a ring extension of the equivariant cohomology of  $G_{2k}(\mathbb{R}^{2n})$ :*

$$\begin{aligned} H_T^*(\tilde{G}_{2k}(\mathbb{R}^{2n})) &\cong \frac{H_T^*(G_{2k}(\mathbb{R}^{2n}))[e^T, \bar{e}^T]}{(e^T)^2 = p_k^T, (\bar{e}^T)^2 = \bar{p}_{n-k}^T, e^T \bar{e}^T = \prod_{i=1}^n \alpha_i} \\ &\cong \frac{\mathbb{Q}[\alpha_1, \alpha_2, \dots, \alpha_n][p_1^T, p_2^T, \dots, p_k^T; \bar{p}_1^T, \bar{p}_2^T, \dots, \bar{p}_{n-k}^T; e^T, \bar{e}^T]}{p^T \bar{p}^T = \prod_{i=1}^n (1 + \alpha_i^2), (e^T)^2 = p_k^T, (\bar{e}^T)^2 = \bar{p}_{n-k}^T, e^T \bar{e}^T = \prod_{i=1}^n \alpha_i} \end{aligned}$$

with additive  $\mathbb{Q}[\alpha_1, \alpha_2, \dots, \alpha_n]$ -basis  $\{(p_1^T)^{r_1} (p_2^T)^{r_2} \cdots (p_k^T)^{r_k} \mid \sum_{i=1}^k r_i \leq n - k\}$  for the  $+1$ -eigenspace of  $\rho^*$  and  $\{e^T \cdot (\bar{p}_1^T)^{r_1} (\bar{p}_2^T)^{r_2} \cdots (\bar{p}_{n-k}^T)^{r_{n-k}} \mid \sum_{i=1}^{n-k} r_i \leq k\} \cup \{\bar{e}^T \cdot (p_1^T)^{r_1} (p_2^T)^{r_2} \cdots (p_k^T)^{r_k} \mid \sum_{i=1}^k r_i \leq n - k\}$  for the  $-1$ -eigenspace.

**6.3.3. Characteristic numbers of orientable Grassmannian.** All the oriented Grassmannians are orientable. Among the real Grassmannians, only  $G_{2k}(\mathbb{R}^{2n})$  and  $G_{2k+1}(\mathbb{R}^{2n+2})$  have nonzero top Betti numbers and hence are orientable. We can integrate equivariant cohomology classes on these Grassmannians using the Atiyah-Bott-Berline-Vergne(ABBV) localization formula 2.2.1. According to the additive  $\mathbb{Q}[\alpha_1, \dots, \alpha_n]$ -module structures of equivariant cohomologies of theses Grassmannians, we shall need to understand the integration of equivariant characteristic classes in various cases for any multi-index  $I = (i_1, \dots, i_k)$ :  $(p^T)^I, e^T \cdot (p^T)^I, \bar{e}^T \cdot (p^T)^I, r^T \cdot (p^T)^I, \tilde{r}^T \cdot (p^T)^I$ .

The equivariant Pontryagin classes of canonical bundles, complementary bundles and tangent bundles are given in Prop 5.2.1. The equivariant Euler classes of canonical bundles and complementary bundles are given in Subsubsection 6.2.1. The  $r^T, \tilde{r}^T$  are given in Theorem 5.1.2 and Prop 6.2.3. In order to apply the ABBV formula, we need a localized expression for the equivariant Euler class of normal bundle at each fixed point or fixed circle.

**Proposition 6.3.2.** *Let  $S$  be a  $k$ -element subset of  $\{1, \dots, n\}$ , the equivariant Euler class of normal bundle at a fixed point or fixed circle associated to  $S, S_{\pm}$  is*

(1) For  $G_{2k}(\mathbb{R}^{2n}), \tilde{G}_{2k}(\mathbb{R}^{2n})$ ,

$$e_{S_{\pm}}^N = e_S^N = \prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)$$

(2) For  $\tilde{G}_{2k}(\mathbb{R}^{2n+1})$ ,

$$e_{S_{\pm}}^N = \pm \prod_{l \in S} \alpha_l \prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)$$

(3) For  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ ,

$$e_{S_{\pm}}^N = \pm \prod_{l \notin S} \alpha_l \prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)$$

(4) for  $G_{2k+1}(\mathbb{R}^{2n+1}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ ,

$$e_S^N = \prod_{l=1}^n \alpha_l \prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)$$

*Proof.* The tangent spaces with weight decomposition at each fixed point or fixed circle of real Grassmannian (hence also oriented Grassmannian) are given in Subsubsection 5.1.2, therefore we get the equivariant Euler classes of normal bundles up to signs as the expressions claimed in current Proposition. To resolve the sign ambiguity, we just need to note that the claimed expressions are invariant under the Weyl groups of the oriented Grassmannians as homogeneous spaces  $G/H$ , and also invariant under the deck transformation  $\rho^*$ .  $\square$

Next, we will compute and relate equivariant characteristic numbers of different Grassmannians.

**Corollary 6.3.6.** *Let  $I = (i_1, \dots, i_k)$  be a multi-index and  $\mathcal{S}$  be the collection of all  $k$ -element subsets of  $\{1, \dots, n\}$ , then*

$$\begin{aligned} \int_{\tilde{G}_{2k}(\mathbb{R}^{2n})} (p^T)^I &= \int_{\tilde{G}_{2k}(\mathbb{R}^{2n+1})} e^T \cdot (p^T)^I = \int_{\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})} \bar{e}^T \cdot (p^T)^I \\ &= 2 \int_{G_{2k}(\mathbb{R}^{2n})} (p^T)^I = 2 \int_{G_{2k+1}(\mathbb{R}^{2n+2})} r^T \cdot (p^T)^I = 2 \int_{\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})} \hat{r}^T \cdot (p^T)^I \\ &= 2 \sum_{S \in \mathcal{S}} \frac{((p_1^T)^{i_1} \cdots (p_k^T)^{i_k})|_S}{\prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)} \end{aligned}$$

*Proof.* When applying the ABBV localization formula 2.2.1, besides the localized Pontryagin classes, we just need to observe that for  $G_{2k}(\mathbb{R}^{2n}), \tilde{G}_{2k}(\mathbb{R}^{2n}), \tilde{G}_{2k}(\mathbb{R}^{2n+1})$  and  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$ , they respectively have

$$\begin{aligned} \frac{e_S^T}{e_S^N} &= \frac{1}{\prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)} & \frac{e_{S_{\pm}}^T}{e_{S_{\pm}}^N} &= \frac{1}{\prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)} \\ \frac{e_{S_{\pm}}^T}{e_{S_{\pm}}^N} &= \frac{\pm \prod_{l \in S} \alpha_l}{\pm \prod_{l \in S} \alpha_l \prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)} & \frac{\bar{e}_{S_{\pm}}^T}{e_{S_{\pm}}^N} &= \frac{\pm \prod_{l \notin S} \alpha_l}{\pm \prod_{l \notin S} \alpha_l \prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)} \end{aligned}$$

for  $G_{2k+1}(\mathbb{R}^{2n+2}), \tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$ , they respectively have

$$\frac{\int r_S^T}{e_S^N} = \frac{\prod_{l=1}^n \alpha_l \int_{S^1} \theta_{S^1}}{\prod_{l=1}^n \alpha_l \prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)} \quad \frac{\int \hat{r}_S^T}{e_S^N} = \frac{\prod_{l=1}^n \alpha_l \int_{\mathbb{R}P^1} \theta_{\mathbb{R}P^1}}{\prod_{l=1}^n \alpha_l \prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)}$$

All these fractions are equal to  $\frac{1}{\prod_{i \in S} \prod_{j \notin S} (\alpha_j^2 - \alpha_i^2)}$ . The difference by factor of 2 comes from the fact that  $\tilde{G}_{2k}(\mathbb{R}^{2n}), \tilde{G}_{2k}(\mathbb{R}^{2n+1})$  and  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})$  have fixed points indexed by  $S_{\pm}$ , while  $G_{2k}(\mathbb{R}^{2n}), G_{2k+1}(\mathbb{R}^{2n+2})$  and  $\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})$  have fixed points or circles indexed by  $S$ .  $\square$

*Remark 6.3.3.* When the cohomological degree of characteristic polynomial matches with the dimension of Grassmannian, or equivalently  $\sum_{j=1}^k j \cdot i_j = k(n-k)$ , we then get a formula of the ordinary characteristic numbers by substituting  $\alpha_i = a_i \in \mathbb{Q}$  such that  $a_i \neq 0, a_i \neq a_j$  into the localized expression of ABBV formula. Moreover, we have the relations between ordinary Pontryagin characteristic numbers:

$$\begin{aligned} \int_{\tilde{G}_{2k}(\mathbb{R}^{2n})} p^I &= \int_{\tilde{G}_{2k}(\mathbb{R}^{2n+1})} e \cdot p^I = \int_{\tilde{G}_{2k+1}(\mathbb{R}^{2n+1})} \bar{e} \cdot p^I \\ &= 2 \int_{G_{2k}(\mathbb{R}^{2n})} p^I = 2 \int_{G_{2k+1}(\mathbb{R}^{2n+2})} r \cdot p^I = 2 \int_{\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})} \tilde{r} \cdot p^I \end{aligned}$$

*Remark 6.3.4.* Recall that localized equivariant Pontryagin class is obtained from localized equivariant Chern class by replacing  $\alpha_i$  with  $\alpha_i^2$ . Let  $Sq : \mathbb{Q}[\alpha_1, \dots, \alpha_n]$  be the ring homomorphism by sending  $\alpha_i$  to  $\alpha_i^2$  for every  $i$ . Then we have

$$\int_{G_{2k}(\mathbb{R}^{2n})} (p^T)^I = Sq \left( \int_{G_k(\mathbb{C}^n)} (c^T)^I \right)$$

Since the ring homomorphism  $Sq$  keeps rational numbers unchanged, we have the relation of ordinary Pontryagin numbers and Chern numbers

$$\int_{G_{2k}(\mathbb{R}^{2n})} p^I = \int_{G_k(\mathbb{C}^n)} c^I$$

*Remark 6.3.5.* Consider the 2-covers of orientable Grassmannians  $\pi : \tilde{G}_{2k}(\mathbb{R}^{2n}) \rightarrow G_{2k}(\mathbb{R}^{2n})$  and  $\pi : \tilde{G}_{2k+1}(\mathbb{R}^{2n+2}) \rightarrow G_{2k+1}(\mathbb{R}^{2n+2})$ . By the naturality of equivariant Pontryagin classes and the above relations among equivariant characteristic numbers, we see

$$\int_{\tilde{G}_{2k}(\mathbb{R}^{2n})} \pi^*((p^T)^I) = \int_{\tilde{G}_{2k}(\mathbb{R}^{2n})} (p^T)^I = 2 \int_{G_{2k}(\mathbb{R}^{2n})} (p^T)^I$$

From Prop 6.2.4, we have  $\pi^*r^T = 2\tilde{r}^T$ , then

$$\int_{\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})} \pi^*(r^T \cdot (p^T)^I) = 2 \int_{\tilde{G}_{2k+1}(\mathbb{R}^{2n+2})} \tilde{r}^T \cdot (p^T)^I = 2 \int_{G_{2k+1}(\mathbb{R}^{2n+2})} r^T \cdot (p^T)^I$$

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